

1. Clearly show your work by explaining how the graph at the top (right) of page 61 can be adjusted to solve these three problems.
 - (a) (West 1.4.26) Arrange seven 0's and seven 1's cyclically so that the 14 strings of four consecutive bits are all the 4-digit binary strings other than 0101 and 1010.
 - (b) Solve the last problem with another pairs of strings (other than 0101 and 1010).
 - (c) Prove that this was the only other pair (besides 0101 and 1010) that you could have removed to obtain a de Bruijn sequence of length 14.

I'll address the last question first. In the proof of Theorem 1.4.26, a correspondence is drawn between Euler circuits on the digraph D_n and de Bruijn sequences. That correspondence still holds when some edges (corresponding to some sequences $a_1 \dots a_n$) are removed. So we seek to remove two edges from D_n leaving a digraph with an Euler circuit. A digraph is Eulerian if and only if the indegree equals the outdegree at each vertex. To maintain this property, we can only remove cycles from D_n to leave an Eulerian graph. The only cycles of length ≤ 2 in the digraph at the top-right of page 61 are the two loops (corresponding to 0000 and 1111) and the cycle in the middle (corresponding to 0101 and 1010). To find the de Bruijn mini-sequences of length 14 requested, remove a pair of edges and locate an Eulerian circuit. The edges along the circuit will be the sequence.

2. (West 1.4.29) Suppose G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle.

Consider each pair v_i, v_{i+1} in an odd cycle v_1, \dots, v_k of G . (For convenience of notation, define $v_0 = v_k$.) Some edges will be directed from v_i to v_{i+1} and some from v_{i+1} to v_i . If the edge was from v_{i+1} to v_i , there must also be a path from v_i to v_{i+1} since D is strongly connected. If, in this latter case, a path from v_{i+1} to v_i of even length, then this path, together with the edge from v_i to v_{i+1} forms an odd cycle, and we're done. So we're left with the case that for all i either there is an edge from v_i to v_{i+1} or an odd path from v_i to v_{i+1} , but in either case we have an odd path. Connecting these k odd paths together, we obtain a closed odd walk. But we've proved that if a graph has a closed odd walk, it has a closed odd cycle (Lemma 1.2.15 on page 24). (The proof given there goes through just fine on digraphs.)

3. A Hamiltonian cycle in a graph is a cycle which contains all the vertices (but not necessarily all the edges) in the graph. Unlike Eulerian tours, there is no known efficient algorithm to detect whether a graph has a Hamiltonian cycle. (See page 286 of West for a longer introduction.)

A Grey code, like a de Bruijn sequence, is a way to cycle through all the k -bit binary strings. Each string can be obtained from the previous by changing one bit. For example, here's a 3-bit Grey code:

$$000 \rightarrow 010 \rightarrow 011 \rightarrow 111 \rightarrow 110 \rightarrow 100 \rightarrow 101 \rightarrow 001 \rightarrow 000$$

(Some background not relevant to the problem) Grey codes are useful in control design because they avoid *race conditions*. If a counter counted normally, going 000, 001, 010, ..., there is an opportunity for a *race condition*. If both bit changes going from 001 to 010 don't happen at *exactly* the same time, then 011 or 000 might appear in between for a brief moment.

- (a) Prove (by induction) that any hypercube Q_k contains a Hamiltonian cycle.

We'll prove that Q_k has a Hamiltonian cycle for $k \geq 2$. Q_k is constructed from two copies Q_{k-1} by connecting corresponding vertices. By induction, each copy of Q_{k-1} has the same Hamiltonian cycle, say v_1, \dots, v_n, v_1 in the first and v'_1, \dots, v'_n, v'_1 in the second. To join the two Hamiltonian cycles into 1, traverse the following edges: $v_1, \dots, v_n, v'_n, \dots, v'_1, v_1$. For the base case, when $k = 2$, Q_k is a square which is a Hamiltonian cycle.

- (b) Use your result to prove that an k -bit Grey code exists for all k .

Recall the construction of a Hypercube Q_k from k -tuples with entries $\{0, 1\}$ (West, page 35). Two k -tuples are connected if they differ in one bit. Hence a Hamiltonian cycle is exactly a Gray code. This proves that Gray codes exist for $k \geq 2$. Constructing a Gray code $k = 1$ is easy; it's just $0 \rightarrow 1 \rightarrow 0$.