MCS-236
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Selected solutions solution

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1. (West 3.1.19) Let $A = (A_1, \ldots, A_m)$ be a collection of subsets of a set Y. A *system of distinct representatives* (SDR) for A is a set of distinct elements $a_1, \ldots a_m$ in Y such that $a_i \in A_i$. Prove that A has an SDR if and only if $|\bigcup_{i \in S}| \geq |S|$ for every $S \subseteq \{1, \ldots, m\}$.

   As an example, the sets $\{a, c\}$, $\{a, d\}$, $\{d, e\}$, and $\{a, b, d\}$ has the SDR $\{a, d, e, b\}$, since the four elements are distinct and $a \in \{a, c\}$, $d \in \{a, d\}$ $e \in \{d, e\}$ and $b \in \{a, b, d\}$.

   On the other hand, $A = \{a, b\}$, $B = \{b, c, d, f\}$, $C = \{a, e\}$, $D = \{b, e\}$, and $E = \{a, b, e\}$ don't because the $A \cup C \cup D \cup E = \{a, b, e\}$ which has only three elements.

   Hint: Transform this to a graph problem.

   Construct a bipartite graph with one vertex for each set A_k and one vertex on the right for each element of y_j of Y. Connect the vertices corresponding to A_k and y_j by an edge whenever $y_j \in A_k$.
   Now a perfect matching in the graph corresponds exactly to an SDR over the $(A_1, \ldots, A_m)$. Further, the condition in Hall's Matching Theorem that $|N(S) \geq |S|$ corresponds exactly to the condition that $|\bigcup_{i \in S}| \geq |S|$. Hence, Hall's Theorem gives the result.