David Wolfe

# 1 Planar graphs, Kuratowski's Theorem and Euler's Formula

A *planar embedding* of a graph is a graph drawn on the plane with no edges crossing. A *planar graph* is a graph which has a planar embedding. For instance, planar graphs depict circuits which can be made on a single layer circuit board.

**Theorem 1 (Euler's Formula)** *Any planar embedding of a connected graph has* $|E| - |V| + 2$ *regions.*

**Proof**: Let $G$ be the graph with the planar embedding. Suppose $G$ has a cycle, then we can remove an edge from $G$, leaving it connected. By induction, the remaining graph has $|E| - |V| + 1$ regions. Adding the edge back in divides one of the regions in two, creating $|E| - |V| + 2$ regions. The base case is when $G$ is a tree, and $|E| = |V| - 1$, and there is one (outside) region. ■

**Corollary 1** *For any connected, planar graph* $|E| \leq 3|V| - 6$

**Proof**: Call $R$ the set of regions of $G$. Let $N_r$ be the number of edge-sides region $r \in R$ touches. (A few regions will touch the same edge on both sides of the edge.) For each region, $N_r \geq 3$, and so $3|R| \leq \sum_{r \in R} N_r = 2|E|$. So, by Euler's formula, $\frac{2}{3}|E| \geq |E| - |V| + 2$. ■

**Corollary 2** *Every planar graph has a vertex,* $v$*, of degree* $d(v) \leq 5$.

**Proof**: If not, $d(v) \geq 6$ for all vertices, and $|E| \geq 3|V|$, contradicting the last corollary. ■

**Corollary 3** *Every planar graph is 6-colorable.*

**Proof**: Let vertex $v$ have degree at most 5. Color the remaining graph by induction, then $v$ can be colored. For the base case, a single vertex graph can be colored trivially. ■

**Corollary 4** *A planar graph is 5-colorable.*

**Proof**: The last proof can be strengthened by a trick. Again, let vertex $v$ have degree at most 5. 5-color the remaining graph by induction. If $v$ is connected to 4 different colors, then $v$ can be colored easily. Otherwise, $v$ is connected to vertices, $v_1, v_2, v_3, v_4, v_5$ listed in clockwise order with colors $1, 2, 3, 4, 5$, respectively. Consider the subgraph consisting of all vertices colored 1 and 3. If the connected component of $v_1$ does no contain $v_3$, we can reverse the colors 1 and 3 in $v_1$'s connected component, leaving a legal coloring. Then, $v$ can be colored 1. If, however, $v_1$ and $v_3$

are not in the same connected component, then consider the subgraph consisting of all vertices colored 2 and 4. In this subgraph, $v_2$ must be disconnected from $v_4$, and again we can reverse the colors in $v_2$'s connected component, and color $v$ with a 2. ∎

**Theorem 2** *A planar graph is 4-colorable.*

This was shown in 1977 by Appel and Haken by enumerating many cases by computer.

Determining if a planar graph is 3-colorability is NP-complete. Determining if a planar graph is 2-colorable is easy:

**Theorem 3** *A graph is 2-colorable if and only if it has no odd cycles.*

**Proof**: The *only if* is trivial. For the *if*, w.l.o.g., assume the graph is connected. Pick a node, $v$, and color it red. All nodes an even distance from $v$ color red, and those an odd distance color blue. If a node, $u$, is both an even distance and an odd distance from $v$, then the graph must have an odd cycle. ∎

Next, without proof, we give a complete characterization of planar graphs known as Kurotowski's Theorem (1930).

A graph $G$ is *homeomorphic* to $H$ if it can be obtained by adding and deleting degree 2 vertices. Adding a vertex consists of placing along an existing edge, and removing it consists of leaving a single edge in place of the two edges leaving the vertex.

**Theorem 4 (Kuratowski's Theorem)** *A graph is planar if and only if it does not contain a graph homeomorphic to $K_{3,3}$ or $K_5$.*