David Wolfe **A brief note on Trees** **October 22, 2002**

Theorem 1 *A tree is a connected acyclic undirected graph with n vertices. The following conditions are all equivalent:*

1. *G is a tree (i.e., G is a connected acyclic graph.)*
2. *There is exactly one path between every pair of nodes in G.*
3. *G is connected with $n-1$ edges.*
4. *G is acyclic with $n-1$ edges.*

Proof:

$1 \Rightarrow$ **2:** Let G be a tree. By definition, it is connected, and so there is at least one path between every pair of vertices. So all we need to show is that there cannot be more than one path between a pair. Suppose, by way of contradiction, there are two paths from u to v. These two paths, say p_1 and p_2, must intersect at least twice (once at u and once at v and perhaps more often) and must separate at least once (since the paths differ). Let x be the last vertex before p_1 and p_2 first separate, and let y be the first vertex thereafter where the paths intersect. A cycle is formed by following p_1 from x to y, and then tracking p_2 backward from y to x.

$2 \Rightarrow$ **3:** Assume there is exactly one path between every pair of nodes in G. By definition, G is then connected. Remove one edge, say $\{u, v\}$, from G. Since there was exactly one path from u to v, removal of this edge must leave G disconnected into 2 components. Each component has at most one path between any pair of vertices in the component, since this was true in G. Hence, by induction, each of the two components has one fewer edges than vertices. Hence G has $n - 2 + 1 = n - 1$ edges. The base case is a single vertex with no edges.

$3 \Rightarrow$ **4:** Let G be a connected graph with exactly $n-1$ edges. We've already seen that a connected graph has at least $n-1$ edges. Hence, removing one edge, $\{u, v\}$, disconnects the graph into two pieces, each of which has one fewer edge than vertex (since each is connected). By induction, each piece is acyclic. Further, there is no path from u to v. Hence adding the edge $\{u, v\}$ back into the graph cannot introduce a cycle, and so G is acyclic.

$4 \Rightarrow$ **1:** It suffices to prove any acyclic graph, H, has $V - E$ connected components. The proof is by induction. Remove one edge, $\{u, v\}$ from H. Since H is acyclic, there was only one path from u to v, and so $H - \{u, v\}$ has one more component than H. By induction, $H - \{u, v\}$ has $V - (E - 1)$ connected components, and so H had $V - E$. The base case is a graph with no edges and $V - 0$ connected components.

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