

1. Squex is a game like Hex is played on a square board. A player makes a turn by placing a checker of her own color on the board. Squares on the board are *adjacent* if they share a side. Black's goal is to connect the top and bottom edges with a path of black checkers, while White wishes to connect the left and right edges with white checkers.

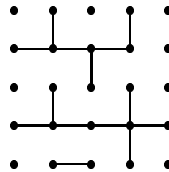
(a) In class, we discussed that the first player should win or draw a Squex position. Briefly describe the proof here.

Suppose it were a win for the second player. The first player adopts player 2's winning strategy  $S$ , but makes an arbitrary first move  $x$ . (The additional move can only help.) Player 1 plays  $S$  as though the  $x$  is not on the board. If strategy  $S$  ever dictates that player 1 should play  $x$ , player 1 makes another arbitrary move, and calls it the new  $x$ . Since strategy  $S$  wins, so does this new adopted strategy, contradicting the fact that the game is a win for player 2.

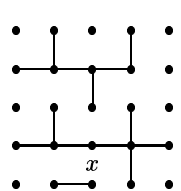
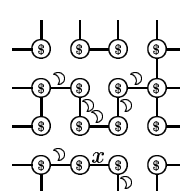
(b) For what values of  $n$  is  $n \times n$  Squex a win for the first player, and when is it a draw? Prove your answer by giving an explicit strategy for the first player to win or the second player to draw as appropriate.

It's a win if and only if  $n = 1$ . On larger boards, suppose player 2 is trying to connect the left side to the right. Player 2 chooses two consecutive rows, and whenever player 1 plays in one of the rows, player 2 plays the same column in the other row. In this way, player 2 can keep player 1 from making any connection joining these two rows, and therefore player 1 cannot make a path from the top to the bottom.

2. (a) Consider the following Dots & Boxes position:



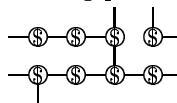
- i. Construct the corresponding Strings & Coins position.
- ii. Determine if the person on move wants an even or odd number of long chains.
- iii. Determine all winning move(s).

There are 24 moves remaining, and 16 boxes. The sum is even, so the person on move wants an odd number of long chains. The large group of coins in the middle will remain one long chain: The only moves which threaten to split it are loony. The group of three coins in the bottom left may become a long chain, and the move at  $x$ , sacrificing two coins, is the only (non-loony) move preventing it and is the only winning move.

(b) Nimstring is the same as Strings & Coins except the first player who cannot move loses. The winner is always the same as in Strings & Coins if there is a sufficiently long chain around.

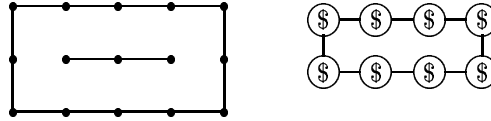
i. Find all winning moves in the following Nimstring position.



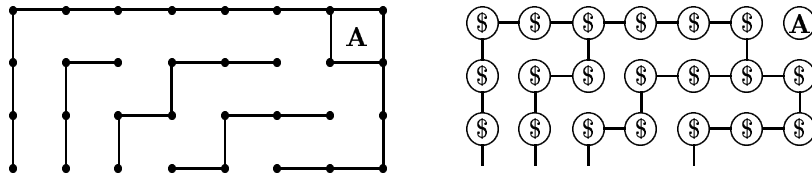
ii. Draw the corresponding Dots & Boxes position. How many boxes will you get in a well played game from this position?

There are 13 moves and 8 coins, so the player on move wants an even number of long chains. Cutting the only vertical string which makes 2 long chains accomplishes this. After any other move, the opponent can guarantee only one long chain, so this is the only winning move. In a Dots & Boxes game, the opponent will get 2 boxes from one of the 2 long chains, and the upper right box, and so you'll win 5 to 3. (By the way, the two pieces have values \*3 and \*1.)

3. There are other Dots & Boxes (or Strings & Coins) positions where every move is loony. For example, there can be cycles:



and many-legged *spiders*:



How can you adapt the Long Chains Theorem and its proof to account for these positions?

The only relevant facts about a long chain in the proof of the Long Chains Theorem were (1) that a long chain only has loony moves, and (2) that the difference in the number of moves and boxes on a long chain is odd. This 2nd point fact is what led to the conclusion that each long chain changes the parity of the number of moves made.

In both cycles and spiders, fact (1) still holds. Each cycle has an equal number of moves and boxes, so counts for an even number of long chains. A spider with  $n$  arms (a long chain has 2 arms) has  $n - 1$  more moves than boxes, and so counts for  $n - 1$  chains.

To see that there are  $n - 1$  more moves than boxes, look at the strings and coins position. Consider the graph where each strings is an edge, and each of the  $B$  coins is a vertex. In addition, define one additional free vertex connected to the end of each arm. The resulting graph is a tree, and so has  $M - 1$  vertices,  $n$  of which are the free vertices, leaving  $B = M - (n - 1)$ .

4. Compose a Dots & Boxes problem and submit it along with a solution. The more challenging, the better, so long as you can analyze it.