

**As always, a proof is required for any problem solution.**

1. Consider the subtraction game where Left is allowed to take any positive odd number of counters and Right may take any positive even number. Determine the outcome classes for the subtraction game on a single heap of  $n \geq 0$  counters.

A heap of size  $n = 0$  is the only  $\mathcal{P}$ -position.

When  $n$  is odd, the heap is an  $\mathcal{L}$ -position. Left wins moving first by taking the whole heap. Right moving first must leave an odd number, and Left again wins by taking the remaining heap.

When  $n$  is even, the heap is an  $\mathcal{N}$ -position. Either player wins in one turn; Left takes  $n - 1$  coins, and Right takes all  $n$  coins.

2. Consider the subtraction game where Left is allowed to take any number of counters which is a power of two  $\{1, 2, 4, 8, \dots\}$ , and Right may take any positive number of counters which is *not* a power of two  $\{3, 5, 6, 7, 9, \dots\}$ . Determine the outcome classes for the subtraction game on a single heap of  $n \geq 0$  counters.

The initial sequence of positions is easily confirmed by hand:

$n$	0	1	2	3	4	5	6	7	8
	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{R}$	$\mathcal{L}$

Fix  $n > 9$ . The heap is an  $\mathcal{N}$ -position if and only if  $n$  can be written as  $2^m + a$  for  $a \in \{0, 1, 2, 4, 8\}$ . All other heaps are  $\mathcal{R}$  positions. To see why, Right can win moving first by moving to 0 or 7, for either  $n$  or  $n - 7$  is not a power of 2. If Left moves first, she wins only by moving to a  $\mathcal{P}$  or  $\mathcal{L}$  position, which, by induction, can happen only when  $n = 2^m + a$  for  $a \in \{0, 1, 2, 4, 8\}$ . In these cases, Left moves immediately to  $a$ .