MCS-344 **Homework set 6 solution** March 13, 2003
David Wolfe Due: March 20, 2003

1. Play a computer program at Dots & Boxes. There are a number of them available on the web, and I've put a link to at least one in my homepage. You should be able to win by both moving first and by moving second. Report your success. In particular, on what size boards you can win against the computer? Is it easy to win? What programs did you play against?

2. Prove, by induction, that any all-small game is infinitesimal.

   Let G be all small, and let $x > 0$ be a number. We'll show that Left wins moving first or second, and hence $G + x > 0$. Hence, any all-small game $-G < x$.
   Left moving first plays on G, and since (by induction) $G^L + x$ is positive, Left wins. In the case that G has no Left option, since the game is all small, neither does Right, and so $G = 0$. Then Left wins since $x > 0$.
   Right moving first can move to $G^R + x$ (also positive, by induction) or $G + x^R$ (positive, by induction, since x^R is a number exceeding x).

3. A clobber *n-snake* has n black stones followed by a white stone. So ●●●●○ is a 4-snake. Prove that an n-snake equals an $(n-1)$-snake plus $\uparrow *$. For example,

   ●●●●●●○ = ●●●●●○ $+ \uparrow *$

   Call denote by x_n a snake with n ●s followed by a single ○. We'll show the second player wins

   $$G = -x_n + x_{n-1} + \uparrow + *.$$

   First, note that $x_n > 0$ when $n \geq 2$. (Left wins moving first or second by taking the sole white stone.) Also, by induction, $-x_{n-1} + x_{n-2} + \uparrow + * = 0$. Using these two facts, we can find a winning response to each option as shown the following tables:

Right moves first	Left's response
$0 + x_{n-1} + \uparrow + *$	$0 + x_{n-1} + \uparrow + 0 > 0$
$-x_n + x_{n-2} + \uparrow + *$	$-x_{n-1} + x_{n-2} + \uparrow + * = 0$
$-x_n + x_{n-1} + * + *$	$-x_{n-1} + x_{n-1} + * + * = 0$
$-x_n + x_{n-1} + \uparrow + 0$	$-x_{n-1} + x_{n-1} + \uparrow + 0 > 0$

Left moves first	Right's response
$-x_{n-1} + x_{n-1} + \uparrow + *$	$-x_{n-1} + x_{n-1} + * + * = 0$
$-x_n + 0 + \uparrow + *$	$-x_n + 0 + * + * < 0$
$-x_n + x_{n-1} + 0 + *$	$-x_n + x_{n-2} + 0 + *$
$-x_n + x_{n-1} + \uparrow + 0$	$-x_n + x_{n-1} + * + 0 < G$

   The last two entries in the second table which require further justification. For the third entry, Left can continue to $-x_{n-1} + x_{n-2} + 0 + *$, whereby Right wins by induction by moving to $-x_{n-1} + x_{n-3} + 0 + *$. For the base case, when $n - 3 = 1$, we have $-x_3 + x_1 + 0 + * = -x_3 < 0$. On the other hand, if Left continues to $-x_{n-1} + x_{n-2} + 0 + 0$, Right can move to $-x_{n-2} + x_{n-2} = 0$ and win.
   To justify the last entry, note that Left's options thereafter are handled by the other cases. Left's position has been made worse the by exchange.
   For the base case, $x_2 - x_1 = \uparrow *$.