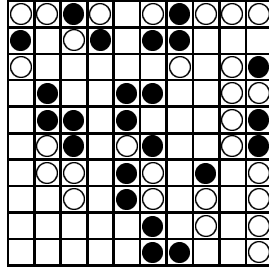


By hook or by crook

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Left to move - can she win?

Abstract

We will show that the above position has a winning move for Left. Although the reader would be hard-pressed to calculate each of the canonical forms, we can accomplish this instead by proving that the total atomic weight exceeds -1 . However, we will go one step further and prove that the overall atomic weight is exactly $-1+2$. We will then demonstrate that the winning move is on $\bullet\bullet\bullet\bullet\circ\circ\circ\circ$ to $\bullet\bullet\bullet\bullet\circ\circ\circ\circ$.

The position above can be broken down into the following sum:

$$\bullet\circ\circ + \bullet\bullet\bullet\circ\circ + \bullet\circ\circ\circ\bullet + \circ\bullet\bullet\circ\circ + \bullet\bullet\bullet\circ\bullet\bullet + \bullet\bullet\bullet\bullet\circ\circ\circ$$

Four of these positions are relatively simple and have canonical forms as follows:

$$\begin{aligned} \bullet\circ\circ &= \downarrow \text{ which has atomic weight } -1, \\ \bullet\bullet\bullet\circ\circ &= *2 \text{ with atomic weight } 0, \\ \circ\bullet\bullet\circ\bullet &= \bullet\circ\circ\bullet\bullet \mid \circ\bullet\circ\circ\bullet, \bullet\circ\circ\circ\bullet = \uparrow * \mid \downarrow, \{ * \mid \Psi * \} \text{ with atomic weight } \downarrow *, \\ \circ\bullet\bullet\circ\circ &= \circ\bullet\bullet \mid \circ\bullet\bullet\circ\circ, \circ\bullet\bullet\circ\circ = \uparrow * \mid \downarrow, \{ 0 \mid \Psi * \} \text{ with atomic weight } \downarrow * \end{aligned}$$

Throughout the remainder of this paper, we will use $aw(g)$ to represent the atomic weight of some game g . To confirm the value of $aw(g)$ for some game g , we will use the fact that

$$\begin{aligned} aw(g) &\geq aw(h) \text{ just if } g > h + \downarrow + \star \text{ and} \\ aw(g) &\leq aw(h) \text{ just if } g < h + \uparrow + \star. \end{aligned}$$

Also, we assert that $\bullet\bullet\bullet\circ\circ = *2$ is sufficient to serve as \star for this position. Additionally, if we let b be the number of black pieces and w be the number of white pieces in some game g , then we assert that $-(w-1) \leq aw(g) \leq b-1$ when $w, b > 0$.

We will show $aw(\bullet\bullet\bullet\circ\circ\bullet\bullet) = 2\uparrow$ by the method described above, i.e.,

$$0 < \bullet\bullet\bullet\circ\circ\bullet\bullet + \{ \uparrow \mid \downarrow, \{ \uparrow \mid \Psi \} \} \mid \Psi + \star \tag{1}$$

$$0 > \bullet\bullet\bullet\circ\circ\bullet\bullet + \{ \uparrow \mid \Psi, \{ \downarrow \mid 5.\downarrow \} \} \mid 5.\downarrow + \star \tag{2}$$

Let $g = \bullet\bullet\bullet\circ\circ\bullet\bullet$ and $h = \{ \uparrow \mid \downarrow, \{ \uparrow \mid \Psi \} \} \mid \Psi$. Also, let $p(g)$ denote an arbitrary position of g excluding g itself. First, we will show that Left wins on $g + h + \star$ moving either first or second. Now, if Left moves first, she can move on g to $\bullet\bullet\bullet\circ\circ\bullet\bullet = \bullet\bullet\bullet\circ\bullet\bullet \mid \bullet\bullet\circ\circ\bullet\bullet = \{ 0 \mid \uparrow *, \uparrow \} \mid \uparrow * = \uparrow$ and $\uparrow + h + \star > 0$.

When Right moves first, he will have to move on g to prevent Left from getting \uparrow , for which Right has no good response. Furthermore, if Left responds to Right's move on g by moving on h , then Right cannot make his next move on g because doing so allows Left to move to \uparrow , and $p(g) + \uparrow + \star > 0$, since $aw(p(g)) \geq -2$.

To complete our proof of (1), we need to find one left response that is sufficient to win regardless of how Right moves. Each row in the table below corresponds one of Right's moves on g , omitting any moves that do not appear promising.

Right moves to	Left's response	Right moves to	Left's response
$\bullet\bullet\bullet\circ\circ\bullet\bullet + h + \star$	$\bullet\bullet\bullet\circ\circ\bullet\bullet + h^L + \star$	$\bullet\bullet\bullet\circ\circ\bullet\bullet + \downarrow + \star$	$\uparrow + \downarrow + \star$
		$\bullet\bullet\bullet\circ\circ\bullet\bullet + \{\uparrow \Downarrow\} + \star$	$\bullet\bullet\bullet\circ\circ\bullet\bullet + \uparrow + \star$
$\bullet\bullet\circ\circ + h + \star$	$\uparrow * + h + \star$	(Note that $\bullet\bullet\circ\circ = \uparrow *$)	
$\bullet\circ\circ\bullet\bullet + h + \star$	$\uparrow * + h + \star$	(Note that $\bullet\circ\circ\bullet\bullet = \uparrow * + *$)	
$\circ\circ\bullet\bullet + h + \star$	$\bullet\circ\bullet\bullet + h + \star$	$(\uparrow + \uparrow *) + \Downarrow + \star$	$* + *$
$\bullet\bullet\circ\circ\circ + h + \star$	$5.\uparrow * + h + \star$	(Note that $\bullet\bullet\circ\circ\circ = \uparrow * + \uparrow$)	
$\circ\circ\circ\bullet\bullet + h + \star$	$\circ\circ\circ\bullet\bullet + h^L + \star$	$\circ\circ\circ\bullet\bullet + \downarrow + \star$	$\circ\circ\circ\bullet\bullet + * + \star$
		$\circ\circ\circ\bullet\bullet + \uparrow \Downarrow + \star$	$\circ\circ\circ\bullet\bullet + \uparrow + \star$
$\circ\circ\bullet\bullet + h + \star$	$\circ\circ\bullet\bullet + h^L + \star$	(See next table)	

[1] $\bullet\bullet\bullet\circ\circ\bullet\bullet = \bullet\bullet\circ\circ\circ\bullet\bullet \mid \bullet\bullet\circ\circ\circ\bullet\bullet, \bullet\bullet\circ\circ\circ\bullet\bullet, \bullet\bullet\circ\circ\circ\bullet\bullet = \uparrow|\uparrow, *, * = \uparrow|\uparrow, *$

[2], [3] $\circ\circ\circ\bullet\bullet = \circ\circ\bullet\bullet \mid \circ\circ\circ\bullet\bullet, \circ\circ\circ\bullet\bullet, \circ\circ\bullet\bullet = \{\uparrow *|\uparrow\}|\uparrow, \uparrow *, \{\uparrow *|0, \uparrow * \} > \star$

From

Right moves to	Left's response	Right moves to	Left's response
$\circ\circ\bullet\bullet + \downarrow + \star$	$\bullet\circ\circ\bullet\bullet + \downarrow + \star$	$(\bullet\circ\circ\bullet\bullet = \uparrow)$	
$\circ\circ\bullet\bullet + \uparrow \Downarrow + \star$	$\bullet\circ\bullet\bullet + \uparrow \Downarrow + \star$	$\uparrow \Downarrow + \star$	$\uparrow + \star$
	$\bullet\circ\bullet\bullet + \Downarrow + \star$	$\uparrow * + \Downarrow + \star$	

Now we will prove (2). Let $h = \{\uparrow|\Downarrow, \{\downarrow|5.\downarrow\}\}|\downarrow$. Right has a winning first move since Left cannot win if Right moves h to $5.\downarrow$. Then, when Left moves first,

Left moves to	Right's response	Left moves to	Right's response
$g + h^L + \star$	$\bullet\circ\circ\bullet\bullet + h^L + \star$	$\bullet\circ\bullet\bullet + h^L + \star$	$\bullet\circ\circ + h^L + \star$
		$\bullet\circ\circ\bullet\bullet + h^L + \star$	$\bullet\circ\circ\bullet\bullet + \Downarrow + \star$
		$\bullet\circ\bullet\bullet + h^L + \star$	$\bullet\circ\bullet\bullet + \downarrow 5.\downarrow + \star$
		$\bullet\circ\bullet\bullet + h^L + \star$	$\bullet\circ\bullet\bullet + \Downarrow + \star$

Thus, $aw(\bullet\bullet\circ\circ\bullet\bullet) = 2\uparrow$.

Next, we will show that the atomic weight of $\bullet\circ\circ\bullet\bullet$ is $-2+2$ by proving the following:

$$0 < \bullet\circ\circ\bullet\bullet + \{9.\uparrow|\uparrow\}|\uparrow + \star \quad (3)$$

$$0 > \bullet\circ\circ\bullet\bullet + \{7.\uparrow|\uparrow\}|\downarrow + \star \quad (4)$$

Let $g = \bullet\circ\circ\bullet\bullet$ and $h = \{9.\uparrow|\uparrow\}|\uparrow$. First, we will prove (3). Note that if Left moves on h , Right has to make his next move on h^L in order to prevent $9.\uparrow$ (since $aw(p(g)) \geq -8$, $p(g) + 9.\uparrow + \star > 0$). Also, if Right moves first on g , we will have Left play on h , forcing Right to respond to \uparrow from h^L .

First, consider all six of Right's moves on g . Observe that Left always has a response to a position with an atomic weight of at least -2. The box indicates where Left will respond and the value to the right of the arrow indicates the value after Left's response.

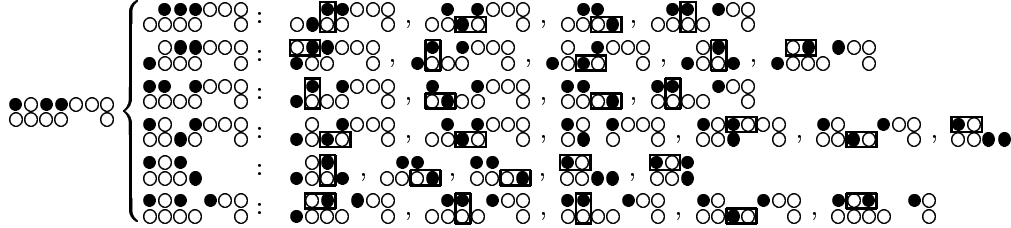
$$\bullet\circ\circ\bullet\bullet \rightarrow *, \bullet\circ\bullet\bullet \rightarrow *, \bullet\circ\bullet\bullet \rightarrow \downarrow *, \bullet\circ\bullet\bullet \rightarrow *, \bullet\circ\bullet\bullet \rightarrow 0, \bullet\circ\bullet\bullet \rightarrow 0$$

If Right first moves on h to \uparrow , Left's response is on g to $\bullet\circ\circ\bullet\bullet + \uparrow + \star$. From here, Right has four possible options on g and one on \uparrow : (i) If Right moves on g to $\bullet\circ\bullet\bullet + \uparrow + \star$, Left responds by moving to $\bullet\circ\bullet\bullet + \uparrow + \star = * + \uparrow + \star \geq 0$. (ii) If Right moves on g to $\bullet\circ\circ\bullet\bullet + \uparrow + \star$, Left replies on g to $\bullet\circ\bullet\bullet + \uparrow + \star = \uparrow + \star \geq 0$. (iii) If Right moves on g to $\bullet\circ\circ\bullet\bullet + \uparrow + \star$, Left can respond by moving on g to $\bullet\circ\bullet\bullet + \uparrow + \star = * + \uparrow + \star \geq 0$. (iv) If Right moves on g to $\bullet\circ\bullet\bullet + \uparrow + \star$, Left wins by moving on g to $\bullet\circ\bullet\bullet + \uparrow + \star \geq 0$. (v) If instead Right moves on \uparrow to $*$, Left responds on g to $\bullet\circ\bullet\bullet = \{\uparrow *|\uparrow\}|\downarrow, \{\uparrow *|\uparrow\}|\downarrow$ and $\{\uparrow *|\uparrow\}|\downarrow, \{\uparrow *|\uparrow\}|\downarrow + * + \star \geq 0$.

Left can win moving first to $g + h^L + \star$, since Right must respond on h^L to \uparrow . Left then responds on g to $\circ\circ\circ\bullet\circ\circ\circ + \uparrow + \star$. Since we proved earlier that $\circ\circ\circ\bullet\circ\circ\circ + \uparrow + \star \geq 0$, we know that $\circ\circ\circ\bullet\circ\circ\circ + \uparrow + \star$ is also positive.

Next, we will prove (4). For this, let $h = \{7.\uparrow|\uparrow\}|\downarrow$. First, we will show that Right has a winning response to every Left first move. If Left starts by moving on h to $g + h^L + \star$, Right responds on g to $\circ\circ\circ\bullet\circ\circ\circ$. We will look at four possible responses by Left (again excluding a symmetrical move). First, Left can move on g to $\bullet\circ\circ\circ\circ\circ$, in which case Right will respond by moving on g to $\bullet\circ\circ\circ\circ\circ$. Left will respond by moving on h^L to $7.\uparrow$ and Right will counter by moving on \star to zero, leaving a total value of zero. Another possible first move for Left is on g to $\circ\circ\circ\bullet\circ$. Right will respond to this by moving on h^L to \uparrow , giving a total value of $\downarrow * + \uparrow + \star$, which is winning for Right. Left could also start by moving on g to $\circ\circ\circ\circ\bullet\circ\circ$. Right again responds by moving on h^L to \uparrow , which makes the total value of the position $\downarrow + \downarrow * + \uparrow + \star$, which is less than zero. If Left moves first on h^L to $7.\uparrow$, Right responds on g to $\circ\circ\circ\bullet\circ\circ\circ$. Since the atomic weight of the resulting position is -1 and a \star is present, we know that it is winning for Right.

If Left instead chooses to first move on g , Right can always respond on h to \downarrow . In this case, Left's only hope is to make two consecutive moves on g . Each of these possibilities is listed below.



Every position listed above has a winning move for Right (indicated by the box), except for $\circ\circ\bullet\circ\circ\circ$, which has a value of zero. Thus, adding \downarrow and \star to any of these positions is winning for Right.

If Right moves first, he will move on g to $\circ\circ\circ\bullet\circ\circ\circ$. Since we already showed that $\circ\circ\circ\bullet\circ\circ\circ + h^L + \star$ wins for Right and since h is better for Right than h^L , we know that $\circ\circ\circ\bullet\circ\circ\circ + h + \star$ is also a win for Right. Thus, $aw(\circ\circ\circ\bullet\circ\circ\circ) = -2 + 2$.

We now know the atomic weight for each of the six original components. The sum of these atomic weights is $-1 + 0 + \downarrow * + \downarrow * + 2\uparrow + -2 + 2 = -1 + 2$, and Left can win moving first. From this, we can see that the only possible winning move for Left must be on $\circ\circ\circ\bullet\circ\circ\circ$.

Our final task is to show Left's winning move for the entire position. To do this, we will show that Left has a move to a position with atomic weight of 1 with \star in it. The move that accomplishes this is the move on $\circ\circ\circ\bullet\circ\circ\circ$ to $\circ\circ\circ\bullet\circ\circ$. We will show that $\circ\circ\circ\bullet\circ\circ$ has atomic weight 0. Again, we will need to prove two inequalities:

$$0 < \circ\circ\circ\bullet\circ\circ\circ + \uparrow + \star \tag{5}$$

$$0 > \circ\circ\circ\bullet\circ\circ\circ + \downarrow + \star \tag{6}$$

In our proof of (3), we showed that Left has a winning response to every Right move on $\circ\circ\circ\bullet\circ\circ\circ + \uparrow + \star$. Now, Left can win moving first on $\circ\circ\circ\bullet\circ\circ\circ$ to $\circ\circ\circ\bullet = \{\uparrow * |\uparrow\} | 0, \{*, \uparrow | *, \downarrow\} + \uparrow + \star$. So, we have completed the proof of (5).

Now, we will prove (6). We have already showed in the proof of (4) that right has a winning response to every left move on $\circ\circ\circ\bullet\circ\circ\circ + \downarrow + \star$. All we have left to show is that Right wins moving first. In this case, Right's best move is to $\circ\circ\circ\bullet\circ\circ\circ + \downarrow + \star = *|\downarrow + \downarrow + \star$, which is winning for Right.

So after Left moves on $\circ\circ\circ\bullet\circ\circ\circ$ to $\circ\circ\circ\bullet\circ\circ$, the atomic weight of the total position is now 1. Since there exists a \star in this position, we know that the value of the total position is at least zero. Hence, a win for Left is assured.

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