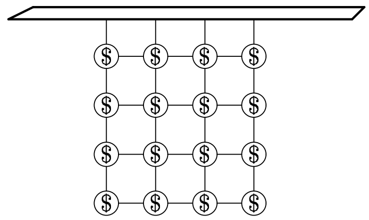
Elwyn Berlekamp, David Wolfe, Aaron Siegel, Adam Duffy, Garrett Kolpin

1. Two players play a vertex deletion game on an undirected graph. A turn consists of removing exactly one vertex of even degree (and all edges incident to it.) Determine under what conditions the first player has a winning strategy and determine the strategy.

   At the end of the game, all vertices are odd degree. Every graph has an even number of odd-degree vertices, so there are an even number of vertices at the end of the game. Hence, the first player wins if and only if the number of vertices in the graph is odd, and the players' moves are irrelevant.

2. A bunch of coins is dangling from the ceiling. The coins are tied to one-another and to the ceiling by strings as pictured below. Players alternately cut strings, and a player whose cut causes any coins to drop to the ground loses. If both players play well, who wins? Prove your answer.



   Let n be the number of coins, and s the number of strings. Each turn, a player must play on a cycle to keep coins from dropping. A graph consisting of the $n+1$ nodes (n coins and 1 ceiling) is connected and has no cycle if and only if it has n edges. (Apply, for instance, Euler's formula with 1 region, 1 connected component.) Hence the game lasts exactly $n-s$ turns.

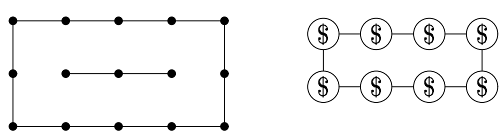
3. The game of *Brussel Sprouts* starts with a number of 4-armed crosses on a piece of paper. A move is to draw a line from one arm to another arm. The line must not cross any other line or any cross. Once the line is drawn, 2 arms are drawn on the line, one on each side of the line. Determine who wins as a function of n, the number of inital crosses, and prove your answer. (Hint: how many moves does the game last?)

   The first player wins if and only if n is odd.
   Each move either connects two connected components, or divides a region in two. In either case, the resulting region(s) have at least one new arm. Hence, at the end of the game, each region has exactly one arm (for otherwise there would still be move(s) available). Also, the number of arms is constant, since each move removes two arms and adds two.
   Now, if there are n brussel sprouts at the start of the game, then there are $4n$ arms throughout the game, and the game ends when there are $4n$ regions. Since each move either connects two components ($n-1$ moves) or creates a region ($4n-1$ *new* regions), we have a total of $n-1+4n-1=5n-2$ moves in the game.

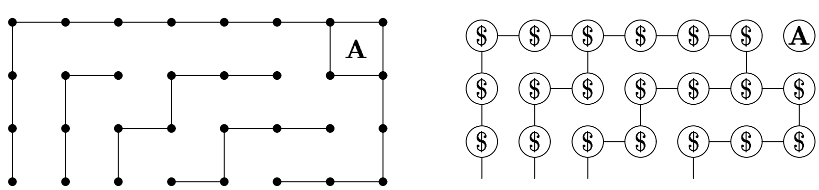
4. Two players play the following game on a round table top of radius R. Players take turns placing pennies (of unit radius) on the tabletop, but no penny is allowed to touch another. Which of the two players has a winning strategy as a function of R? Prove your answer by exhibiting the winning strategy.

   The first player has a win so long as $R \geq 1$. The first player places a coin in the center of the table, and plays 180° symmetry thereafter.

5. There are other Dots & Boxes (or Strings & Coins) positions where every move is loony. For example, there can be cycles:



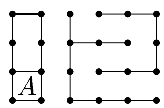
   and many-legged *spiders*:

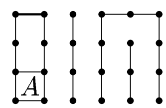


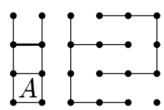
How can you adapt the Long Chains Theorem and its proof to account for these positions?

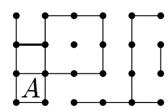
The only relevant facts about a long chain in the proof of the Long Chains Theorem were (1) that a long chain only has loony moves, and (2) that the difference in the number of moves and boxes on a long chain is odd. This second point is what led to the conclusion that each long chain changes the parity of the number of moves made.
In both cycles and spiders, fact (1) still holds. Each cycle has an equal number of moves and boxes, so counts for an even number of long chains. A spider with n arms (a long chain has 2 arms) has $n-1$ more moves than boxes, and so counts for $n-1$ chains.
To see that there are $n-1$ more moves than boxes, look at the strings and coins position. Consider the graph where each strings is an edge, and each of the B coins is a vertex. In addition, define one additional free vertex connected to the end of each arm. The resulting graph is a tree, and so has $M-1$ vertices, n of which are the free vertices, leaving $B = M-(n-1)$.

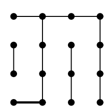
6. In each of the following Dots & Boxes positions, you are A and your opponent is B. It is your turn. Find a winning move if you can. If there are none, write *resign*. Be prepared with a brief explanation. If the score looks close, be sure winning on the long chains is enough!

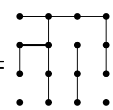
$A =$ 

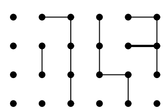
$B =$ 

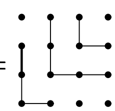
$C =$ 

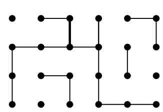
$D =$ 

$E =$ 

$F =$ 

$G =$ 

$H =$ 

$I =$ 

A: 11-4. B: 9-6. C: 13-2. D: 9-6. E: 7-2. F: 6-3.
For G, the positions have value $*3$, $*2$ and $*3$. It requires care to keep high score. Any of the 4 moves near upper right corner are ok. A barely wins 8-7.
For H, player B keeps control only by sacrificing 2. A wins 5-4.
For I, the positions have value $*2$, $*$, and 0. (The second player can force a long chain on the rightmost position.) A wins by sacrificing 2 changing $*2$ to $*$. A squeaks by 8-7.