MAA Summer Seminar **Problem Set 2 solution** August 12, 2003

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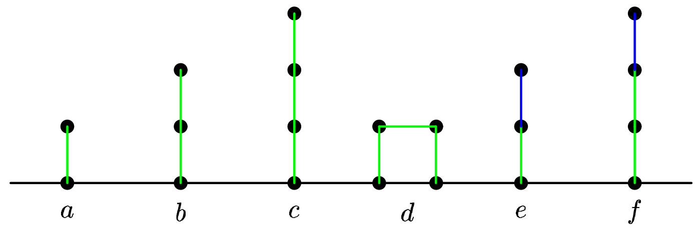
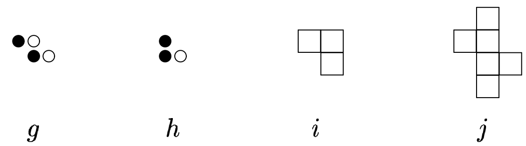
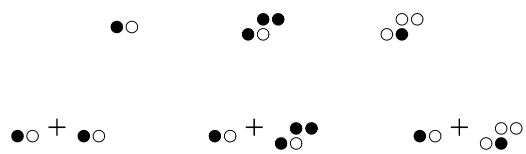
1. Draw a partial order of the following Hackenbush, Clobber and Domineering positions. (Note that some of the positions are equivalent.)





2. Determine whether each of the following six Clobber positions is a win for Left, Right, first or second.



What can we say about sums of games that are first-player wins? Can you find two first-player wins (not necessarily Clobber positions) whose sum is still a first-player win? (Hint: Consider a 2×2 Domineering square in conjunction with one of the Clobber positions above.)

3. Fix a particular (partial) hackenbush position G, with one edge e unspecified. Four different positions can be obtained depending on whether edge e is blue, green, red or missing. (If missing, note that other edges might be disconnected from the ground and therefore removed as well.) Give the partial order on these four games, and prove that the partial order is independent of the position. As always, induction is prefered over arguments like, "and then right continues to..."

Denote the four games by G_+, G_*, G_-, and G_0, respectively. I assert that $G_+ > G_* > G_-$, and $G_+ > G_0 > G_-$, but that $G_* \parallel G_0$. By symmetry, it suffices to show $G_+ > G_*$, $G_+ > G_0$ and that $G_* \parallel G_0$.

- $G_+ - G_* > 0$: Left moving first can remove green edge e in $-G_*$, leaving $G_+ - G_0 > 0$ by induction. If Right removes e in $-G_*$, Right also loses by induction. For any other Right move, write can mirror the move in the other game, leaving some $G'_+ - G'_*$. If the move also removes e, the games are identical, and the difference is 0. If it does not, then $G'_+ - G'_* > 0$ by induction. Either way, Left wins.
- $G_+ - G_0 > 0$: Left moving first can remove the blue edge e in G_+ leaving $G_0 - G_0 = 0$. For Right's move, the argument is the same as the last case above.
- $G_* - G_0 \parallel 0$: Either player can win moving first by removing the green edge in G_* moving to $G_0 - G_0 = 0$.