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1. Consider the following 2-player subtraction game. There are several piles of beans. Each player takes turn moving on one of the piles. A move on a pile of size n consists of removing any proper factor of n beans from the pile. (For example, if there are $n = 12$ beans in a pile, you can leave the pile with 11, 10, 9, 8 or 6 beans.) It's not legal to take the last bean in a pile. The last player to move wins.

   (a) Determine a rule for computing the nim-value of a single pile. Can you prove your rule always works?

   (b) Use your rule to find all winning moves in the four pile game with piles of sizes 8, 10, 12 and 16.

   (c) What happens if any you can remove any factor (not just *proper factors*), so that it's now legal to take a whole pile?

> (a) A pile of size $n = 2^j k$ for k odd has value $*j$. To prove this, we wish to show that (1) the mex of the followers is exactly j, i.e., that there are moves to positions with exponent $j' < j$, and that (2) it's impossible to move to j.
> To show (1), one can remove $2^{j'}k$ coins, for any $j' < j$, leaving $n' = (2^j - 2^{j'})k = 2^{j'}(2j - j' - 1)k$ which is of the form $2^{j'}k'$ where $k' = (2j - j' - 1)k$ is odd.
> To show (2), consider any move to a pile of size $2^j k'$ for $k' < k$ odd. This requires mover to remove $2^j(k - k')$ coins, but then $k - k'$ is even and so $2^j(k - k')$ doesn't divide $2^j k$.
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> (b) The piles have values $*3$, $*1$, $*2$ and $*4$. The unique winning moves is to remove 1 from the pile of size 16, converting the $*4$ to 0.
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> (c) In this case, a pile of size $n = 2^j k$ for k odd has value $*(j + 1)$. The four piles now have values $*4$, $*2$, $*3$ and $*5$, which add to zero, so there are no winning moves.

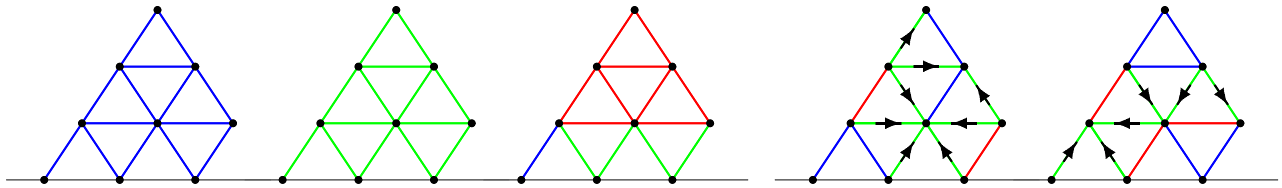
2. Fix a particular (partial) hackenbush position G, with one edge e unspecified. Four different positions can be obtained depending on whether edge e is blue, green, red or missing. (If missing, note that other edges might be disconnected from the ground and therefore removed as well.) Give the partial order on these four games, and prove that the partial order is independent of the position. As always, induction is prefered over arguments like, "and then right continues to..."

> Denote the four games by G_+, G_*, G_-, and G_0, respectively. I assert that $G_+ > G_* > G_-$, and $G_+ > G_0 > G_-$, but that $G_* \parallel G_0$. By symmetry, it suffices to show $G_+ > G_*$, $G_+ > G_0$ and that $G_* \parallel G_0$.
>
> - $G_+ - G_* > 0$: Left moving first can remove green edge e in $-G_*$, leaving $G_+ - G_0 > 0$ by induction. If Right removes e in $-G_*$, Right also loses by induction. For any other Right move, write can mirror the move in the other game, leaving some $G'_+ - G'_*$. If the move also removes e, the games are identical, and the difference is 0. If it does not, then $G'_+ - G'_* > 0$ by induction. Either way, Left wins.
> - $G_+ - G_0 > 0$: Left moving first can remove the blue edge e in G_+ leaving $G_0 - G_0 = 0$. For Right's move, the argument is the same as the last case above.
> - $G_* - G_0 \parallel 0$: Either player can win moving first by removing the green edge in G_* moving to $G_0 - G_0 = 0$.

3. Prove that any red-blue hackenbush position is a number.

> Fix a particular (partial) hackenbush position G, with one edge e unspecified. Three different positions can be obtained depending on whether edge e is blue, missing, or red. Define these games to be G_+, G_0, and G_-, respectively. It's easy to prove that $G_+ > G_0 > G_-$ by playing difference games.
> Now, let H be any blue-red hackenbush position. By the above assertion, every $H^L < H$ and every $H^R > H$. By induction, all H^L and H^R are numbers, so H is the simplest number between all the H^L's and H^R's and is a number.

4. Here is a not-yet-colored Hackenbush graph with 14 branches:



(a) What is the value of the graph if it is colored all blue?

(b) What is the value of the graph if it is colored all green?

(c) Using red, blue, and green colors however you wish, color the branches to make the value of this graph the smallest positive number that you can.

(d) the largest infinitesimal that you can.

(e) the positive infinitesimal of smallest atomic weight that you can.

You need not find the optimal answers.

(a) 14

(b) 0

(c) 2^{-9}:

(d) Atomic weight is 8:

(e) Positive, and atomic weight 0: