Distinguishing Gamblers from Investors at the Blackjack Table

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OUTLINE

• A little history
• Blackjack rules and about card-counting
• Evaluating an individual’s strategy
• Evaluating expected winnings for a baseline strategy
• Unresolved issues
HISTORY

• (1956) Baldwin, Cantey, Maisel, McDermott: Basic strategy
• (1966) Thorp, *Beat the Dealer*
• (1979-1999) Griffin, *The Theory of Blackjack*
• About blackjack card-counters
BLACKJACK RULES

- Play for 21: *hit, stand, bust, hard, soft, deck, shoe*
- Fixed dealer strategy
- Blackjack pays 3:2
- Double down
- Split
- Variations: decks, surrender, insurance, resplitting, double after split, playing multiple hands, ...
## Basic strategy for an infinite deck

<table>
<thead>
<tr>
<th>HARD</th>
<th>SOFT</th>
<th>SPLIT</th>
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<tbody>
<tr>
<td>4–8 HHHHHHHHHH</td>
<td>2 HHHHHHHHHH</td>
<td>A PPPPPPPPPP</td>
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<td>2 HPPPPPPP HH</td>
</tr>
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<td>9 SSSSSSSSS</td>
<td>8 PPPPPPPP PP</td>
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<tr>
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<td>10 SSSSSSSSS</td>
<td>9 SPPPPSPPS</td>
</tr>
<tr>
<td>17–21 SSSSSSSSSS</td>
<td>11 SSSSSSSSSS</td>
<td>T SSSSSSSSSS</td>
</tr>
</tbody>
</table>
CARD COUNTING

• A typical counting system maintains

\[
\frac{\text{(number of 10’s and A’s)}}{\text{(number of 2’s through 6’s)}} - \frac{\text{(number of decks remaining)}}{}
\]

• Effect on play

• Effect on wagers

• Basic strategy gives typically gives dealer .5% advantage

• Counting well typically gives player .5% advantage
GOAL and MOTIVATION

Evaluate a player’s skill, measured by long term expected winnings

- To help gamblers
- Larger question of evaluating decisions under uncertainty
CHALLENGES

- The problems of variance
- No knowledge of player strategy
- Can’t expose player to all possible situations
- Computational efficiency
KEY IDEA

1. Determine and evaluate baseline strategy
2. Compare expectation of player’s choices to baseline
3. Credit or debit player’s expected winnings rather than actual
Define
\[ C \overset{\text{def}}{=} \text{Sample space of cards remaining} \]
\[ O \overset{\text{def}}{=} \text{Sample space specifying ordering of the deck} \]
\[ W_S \overset{\text{def}}{=} \text{Winnings playing strategy } S \]
\[ B \overset{\text{def}}{=} \text{Any automated baseline strategy} \]

Then,
\[ \mathbb{E}_C \{ \mathbb{E}_O \{ W_S \} \} = \mathbb{E}_C \{ \mathbb{E}_O \{ W_B \} \} \]
\[ + \mathbb{E}_C \{ \mathbb{E}_O \{ W_S \} - \mathbb{E}_O \{ W_B \} \} \]

- \( \mathbb{E}_O \{ W \} \) has variance exceeding 1.3 (with unit bets).
- \( \mathbb{E}_O \{ W_S \} - \mathbb{E}_O \{ W_B \} \) typically has variance between \( 10^{-6} \) and \( 10^{-5} \).
\( C \overset{\text{def}}{=} \) Sample space of cards remaining
\( D \overset{\text{def}}{=} \) Sample space specifying ordering of initial deals
\( O \overset{\text{def}}{=} \) Sample space specifying ordering of the deck after deal
\( W_{SS'} \overset{\text{def}}{=} \) Winnings wagering strategy \( S \), playing strategy \( S' \)
\( B \overset{\text{def}}{=} \) Any automated baseline strategy

\[
\mathbb{E}_C \{ \mathbb{E}_D \{ \mathbb{E}_O \{ W_{SS} \} \} \} = \mathbb{E}_C \left\{ \begin{array}{l}
\mathbb{E}_D \{ \mathbb{E}_O \{ W_{SB} \} \} \\
+ \mathbb{E}_D \{ \mathbb{E}_O \{ W_{SS} \} - \mathbb{E}_O \{ W_{SB} \} \}
\end{array} \right\}
\]

- \( \mathbb{E}_D \{ \mathbb{E}_O \{ W_{SB} \} \} \) typically has variance between \( 10^{-3} \) and \( 10^{-4} \).
PERFECT DECISIONS (with replacement)\textsuperscript{a}

\[ p_i \overset{\text{def}}{=} \mathbb{P} \{ \text{drawing card } i \text{ from deck} \} \quad \text{Ace is 1} \]

\[ P^S[i, j] \overset{\text{def}}{=} \mathbb{P} \{ \text{dealer reaches } i \mid \text{starts with soft } j \} \]

\[ P^S[i, j] = \begin{cases} 
1 & \text{if } i = j \geq 17 \\
1 & \text{if } i = j = \text{bust} \\
0 & \text{if } i \neq j \geq 17 \\
P^S[i, j + 10] & \text{if } 7 \leq j \leq 11 \\
\sum_k p_k P^S[i, j + k] & \text{otherwise}
\end{cases} \]

\[
p_i \overset{\text{def}}{=} \mathbb{P} \{\text{drawing card } i \text{ from deck}\} \quad \text{Ace is 1}
\]
\[
P^S[i, j] \overset{\text{def}}{=} \mathbb{P} \{\text{dealer reaches } i \mid \text{starts with soft } j\}
\]
\[
P^H[i, j] \overset{\text{def}}{=} \mathbb{P} \{\text{dealer reaches } i \mid \text{starts with hard } j\}
\]
\[
= p_1 P^S[i, j + 1] + \sum_{2 \leq k \leq 10} p_k P^H[i, j + 10]
\]
\[
P[i, j] \overset{\text{def}}{=} \mathbb{P} \{\text{dealer reaches } i \mid \text{starts with card } j\}
\]

**conditioning on dealer not having blackjack**

\[
= \begin{cases} 
    P^H[i, j] & \text{if } 2 \leq j \leq 9 \\
    (P^H[21, 10] - p_1)/(1 - p_1) & \text{if } i = 21 \text{ and } j = 10 \\
    P^H[i, 10]/(1 - p_1) & \text{if } i \neq 21 \text{ and } j = 10 \\
    (P^S[21, 1] - p_{10}/(1 - p_{10}) & \text{if } i = 21 \text{ and } j = 1 \\
    P^S[i, 1]/(1 - p_{10}) & \text{if } i \neq 21 \text{ and } j = 1
\end{cases}
\]
\[
P[i, j] \overset{\text{def}}{=} \mathbb{P} \{ \text{dealer reaches } i \mid \text{starts with card } j \}
\]

<table>
<thead>
<tr>
<th>(j)</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
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<td>0.189</td>
<td>0.189</td>
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<td>2</td>
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<td>0.121</td>
<td>0.371</td>
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\[
\begin{align*}
    p_i & \overset{\text{def}}{=} \mathbb{P} \{ \text{drawing card } i \text{ from deck} \} \quad \text{Ace is 1} \\
    P[i, j] & \overset{\text{def}}{=} \mathbb{P} \{ \text{dealer reaches } i \mid \text{starts with card } j \} \\
    E_a^H[x, y] & \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{player hard } y, \text{action } a \} \\
    E_a^S[x, y] & \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{player soft } y, \text{action } a \} \\
    a & \in \{ (h)it, (s)tand, (d)ouble down, s(p)lit \} \\
    E_s^H[x, y] & = P[\text{bust}, x] + \sum_{17 \leq k \leq y-1} P[k, x] - \sum_{y+1 \leq k \leq 21} P[k, x] \\
    E_s^S[x, y] & = \begin{cases} 
        E_s^H[x, y] & \text{if } y \geq 12 \\
        E_s^H[x, y + 10] & \text{if } y \leq 11 
    \end{cases}
\end{align*}
\]
\[ E_s^H[x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{stand with } y \text{ versus dealer } x \} \]

<table>
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<tr>
<th>y</th>
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<td>.892</td>
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<td>.926</td>
<td>.931</td>
<td>.939</td>
<td>.963</td>
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</tbody>
</table>
$$E_a^H[x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{player hard } y, \text{action } a \}$$

$$E_a^S[x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{player soft } y, \text{action } a \}$$

$$E_{sh}^H[x, y] \overset{\text{def}}{=} \max E_s[x, y], E_h[x, y]$$

$$E_{sh}^S[x, y] \overset{\text{def}}{=} \max E_s[x, y], E_h[x, y]$$

$$E^S_h[x, y] = \sum_{1 \leq k \leq 10} p(k) E_{sh}^S[x, y + k]$$

$$E^H_h[x, y] = p(1) E_{sh}^S[x, y + 1] + \sum_{2 \leq k \leq 10} p(k) E_{sh}^H[x, y + k]$$
$E_a^H [x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{ player hard } y, \text{ action } a \}$

$E_a^S [x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dealer } x, \text{ player soft } y, \text{ action } a \}$

*after doubling down, you must take exactly one card*

$E_d^H [x, y] = p(1)E_s^S [x, y + 1] + \sum_{2 \leq k \leq 10} p(k)E_s^H [x, y + k]$

$E_d^H [x, y] = p(1)E_s^S [x, y + 1] + \sum_{2 \leq k \leq 10} p(k)E_s^H [x, y + k]$
$E^H_a[x, y] \overset{\text{def}}{=} E \{\text{winnings} \mid \text{dealer } x, \text{player hard } y, \text{action } a\}$

$E^S_a[x, y] \overset{\text{def}}{=} E \{\text{winnings} \mid \text{dealer } x, \text{player soft } y, \text{action } a\}$

*You must take exactly one card on each split ace*

$E_p[x, 2] = 2 \cdot \sum_{1 \leq k \leq 10} E^S_s[x, x + k]$

*Resplitting other cards is possible*

$E_p[x, y] = 2 \cdot (p(y/2)E_p[x, y] + p(1)E^S_{hsd}[x, y/2 + 1]$

$\quad + \sum_{\substack{k \leq 2 \leq 10 \\ k \neq y/2}} E^H_{hsd}[x, y/2 + k])$

$2 \cdot (p(1)E^S_{hsd}[x, y/2 + 1] + \sum_{\substack{k \leq 2 \leq 10 \\ k \neq y/2}} E^H_{hsd}[x, y/2 + k])$

$E_p[x, y] = \frac{1 - 2p(y/2)}{1 - 2p(y/2)}$

*but $E_p[x, y]$ value could be infinite...*
\[ E[x, y] \overset{\text{def}}{=} \mathbb{E} \{ \text{winnings} \mid \text{dlr } x, \text{ plr } y, \text{ dlr has no blackjack} \} \]

\[ E[10, y] = (1 - p(1))E_{\text{legal}}[10, y] - p(1) \]

\[ E[1, y] = (1 - p(10))E_{\text{legal}}[1, y] - p(10) \]

\[ E[x, y] = (1 - p(10))E_{\text{legal}}[1, y] - p(10) \]

*Player’s blackjack plays 3 : 2, unless dealer also has blackjack*

\[ E[10, bj] = 3(1 - p(1))/2 \]

\[ E[1, bj] = 3(1 - p(10))/2 \]

\[ E[x, bj] = 3/2 \]
HIGH-LOW COUNT SYSTEMS

- Count \( \text{Tens + Aces - 2s - 3s - 4s - 5s - 6s} \)
- Balanced count system
- Count per deck affects both play and wager strategy
- Wagers vary from 1 unit to 10 units
- Dubner 1963, Wong 1994
- Wong predicts expected winnings of \(.020 \pm .002\) units per hand

\[
S = \text{Wong’s strategy} \\
B = \text{semi-perfect strategy}
\]
## SIMULATION RESULTS

<table>
<thead>
<tr>
<th>Hands Simulated</th>
<th>Average Actual Winnings</th>
<th>Hands</th>
<th>Assessed Expectation</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Run 1</td>
<td>Run 2</td>
<td>Run 1</td>
</tr>
<tr>
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</tr>
<tr>
<td>Variance</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Difference of about .04 due to approximations of EV’s
UNRESOLVED ISSUES

- Assessing risk: \( E \{\log(\$)\} \)
- Why not change the distribution to assess?
- Playing to affect the deck
- Real casino: deck isn’t random, multiple players, multiple hands, etc.