Distinguishing Gamblers from Investors at the Blackjack Table

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OUTLINE

- A little history
- Blackjack rules and about card-counting
- Evaluating an individual's strategy
- Evaluating expected winnings for a baseline strategy
- Unresolved issues

HISTORY

- (1956) Baldwin, Cantey, Maisel, McDermott: Basic strategy
- (1966) Thorp, *Beat the Dealer*
- (1979-1999) Griffin, *The Theory of Blackjack*
- About blackjack card-counters

BLACKJACK RULES

- Play for 21: *hit, stand, bust, hard, soft, deck, shoe*
- Fixed dealer strategy
- Blackjack pays 3:2
- Double down
- Split
- Variations: decks, surrender, insurance, resplitting, double after split, playing multiple hands, ...

Basic strategy for an infinite deck

HARD		SOFT		SPLIT	
4-8	HHHHHHHHHH	2	HHHHHHHHHH	A	PPPPPPPPPP
9	HHDDDDHHHH	3	HHHHHDHHHH	2	HPPPPPPHHH
10	HDDDDDDDDH	4	HHHHDDHHHH	3	HPPPPPPHHH
11	HDDDDDDDDD	5	HHHHDDHHHH	4	HHHHPPHHHH
12	HHHSSSHHHH	6	HHHDDDHHHH	5	HDDDDDDDDH
13	HSSSSSHHHH	7	HHDDDDHHHH	6	HPPPPPHHHH
14	HSSSSSHHHH	8	HSDDDDSSHH	7	HPPPPPPHHH
15	HSSSSSHHHH	9	SSSSSSSSSS	8	PPPPPPPPPP
16	HSSSSSHHHH	10	SSSSSSSSSS	9	SPPPPPSPPS
17-21	SSSSSSSSSS	11	SSSSSSSSSS	T	SSSSSSSSSS

CARD COUNTING

- A typical counting system maintains

$$\frac{\text{(number of 10's and A's)} - \text{(number of 2's through 6's)}}{\text{(number of decks remaining)}}$$

- Effect on play
- Effect on wagers
- Basic strategy gives typically gives dealer .5% advantage
- Counting well typically gives player .5% advantage

GOAL and MOTIVATION

Evaluate a player's skill, measured by long term expected winnings

- To help gamblers
- Larger question of evaluating decisions under uncertainty

CHALLENGES

- The problems of variance
- No knowledge of player strategy
- Can't expose player to all possible situations
- Computational efficiency

KEY IDEA

1. Determine and evaluate baseline strategy
2. Compare expectation of player's choices to baseline
3. Credit or debit player's expected winnings rather than actual

Define

$$
\begin{aligned}
C &\stackrel{\text{def}}{=} \text{Sample space of cards remaining} \\
O &\stackrel{\text{def}}{=} \text{Sample space specifying ordering of the deck} \\
W_S &\stackrel{\text{def}}{=} \text{Winnings playing strategy } S \\
B &\stackrel{\text{def}}{=} \text{Any automated baseline strategy}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathbf{E}_C\{\mathbf{E}_O\{W_S\}\} &= \mathbf{E}_C\{\mathbf{E}_O\{W_B\}\} \\
&\quad + \mathbf{E}_C\{\mathbf{E}_O\{W_S\} - \mathbf{E}_O\{W_B\}\}
\end{aligned}
$$

- $\mathbf{E}_O\{W\}$ has variance exceeding 1.3 (with unit bets).
- $\mathbf{E}_O\{W_S\} - \mathbf{E}_O\{W_B\}$ typically has variance between 10^{-6} and 10^{-5}.

$$
\begin{aligned}
C &\stackrel{\text{def}}{=} \text{Sample space of cards remaining} \\
D &\stackrel{\text{def}}{=} \text{Sample space specifying ordering of initial deals} \\
O &\stackrel{\text{def}}{=} \text{Sample space specifying ordering of the deck after deal} \\
W_{SS'} &\stackrel{\text{def}}{=} \text{Winnings wagering strategy } S\text{, playing strategy } S' \\
B &\stackrel{\text{def}}{=} \text{Any automated baseline strategy}
\end{aligned}
$$

$$
\mathbf{E}_C \left\{ \mathbf{E}_D \left\{ \mathbf{E}_O \left\{ W_{SS} \right\} \right\} \right\} = \mathbf{E}_C \left\{ \begin{array}{l} \mathbf{E}_D \left\{ \mathbf{E}_O \left\{ W_{SB} \right\} \right\} \\ + \, \mathbf{E}_D \left\{ \mathbf{E}_O \left\{ W_{SS} \right\} - \mathbf{E}_O \left\{ W_{SB} \right\} \right\} \end{array} \right\}
$$

- $\mathbf{E}_D \left\{ \mathbf{E}_O \left\{ W_{SB} \right\} \right\}$ typically has variance between 10^{-3} and 10^{-4}.

PERFECT DECISIONS (with replacement)[a]

$$p_i \stackrel{\text{def}}{=} \mathbf{P}\{\text{drawing card } i \text{ from deck}\} \quad \text{Ace is 1}$$

$$P^S[i,j] \stackrel{\text{def}}{=} \mathbf{P}\{\text{dealer reaches } i \mid \text{starts with soft } j\}$$

$$P^S[i,j] = \begin{cases} 1 & \text{if } i = j \geq 17 \\ 1 & \text{if } i = j = \text{bust} \\ 0 & \text{if } i \neq j \geq 17 \\ P^S[i, j+10] & \text{if } 7 \leq j \leq 11 \\ \sum_k p_k P^S[i, j+k] & \text{otherwise} \end{cases}$$

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[a] Art Benjamin, Eric Huggins. *Optimal Blackjack Strategy with "Lucky Bucks."* UMAP **14** 4 (1993).

$$p_i \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{drawing card } i \text{ from deck}\right\} \quad \text{Ace is 1}$$

$$P^S[i,j] \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{dealer reaches } i \mid \text{starts with soft } j\right\}$$

$$P^H[i,j] \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{dealer reaches } i \mid \text{starts with hard } j\right\}$$

$$= p_1 P^S[i, j+1] + \sum_{2 \le k \le 10} p_k P^H[i, j+10]$$

$$P[i,j] \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{dealer reaches } i \mid \text{starts with card j}\right\}$$

conditioning on dealer not having blackjack

$$= \begin{cases} P^H[i,j] & \text{if } 2 \le j \le 9 \\ (P^H[21,10] - p_1)/(1-p_1) & \text{if } i = 21 \text{ and } j = 10 \\ P^H[i,10]/(1-p_1) & \text{if } i \ne 21 \text{ and } j = 10 \\ (P^S[21,1] - p_{10}/(1-p_{10}) & \text{if } i = 21 \text{ and } j = 1 \\ P^S[i,1]/(1-p_{10}) & \text{if } i \ne 21 \text{ and } j = 1 \end{cases}$$

$$P[i,j] \stackrel{\text{def}}{=} \mathbf{P}\{\text{dealer reaches } i \mid \text{starts with card j}\}$$

j	17	18	19	20	21	22
1	0.189	0.189	0.189	0.189	0.078	0.167
2	0.140	0.135	0.130	0.124	0.118	0.354
3	0.135	0.130	0.126	0.120	0.115	0.374
4	0.130	0.126	0.121	0.116	0.111	0.394
5	0.122	0.122	0.118	0.113	0.108	0.416
6	0.165	0.106	0.106	0.102	0.097	0.423
7	0.369	0.138	0.079	0.079	0.074	0.262
8	0.129	0.359	0.129	0.069	0.069	0.245
9	0.120	0.120	0.351	0.120	0.061	0.228
10	0.121	0.121	0.121	0.371	0.037	0.230

$$p_i \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{drawing card } i \text{ from deck}\right\} \quad \text{Ace is 1}$$

$$P[i,j] \stackrel{\text{def}}{=} \mathbf{P}\left\{\text{dealer reaches } i \mid \text{starts with } \textbf{card } j\right\}$$

$$E_a^H[x,y] \stackrel{\text{def}}{=} \mathbf{E}\left\{\text{winnings} \mid \text{dealer } x\text{, player hard } y\text{, action } a\right\}$$

$$E_a^S[x,y] \stackrel{\text{def}}{=} \mathbf{E}\left\{\text{winnings} \mid \text{dealer } x\text{, player soft } y\text{, action } a\right\}$$

$$a \in \{(h)\text{it, } (s)\text{tand, } (d)\text{ouble down, s}(p)\text{lit}\}$$

$$E_s^H[x,y] = P[\text{bust}, x] + \sum_{17 \leq k \leq y-1} P[k,x] - \sum_{y+1 \leq k \leq 21} P[k,x]$$

$$E_s^S[x,y] = \begin{cases} E_s^H[x,y] & \text{if } y \geq 12 \\ E_s^H[x,y+10] & \text{if } y \leq 11 \end{cases}$$

$$E_s^H[x,y] \stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{stand with } y \text{ versus dealer } x\}$$

y	1	2	3	4	5	6	7	8	9	10
16	-.667	-.293	-.252	-.211	-.167	-.154	-.475	-.511	-.543	-.540
17	-.478	-.153	-.117	-.081	-.045	.012	-.107	-.382	-.423	-.420
18	-.100	.122	.148	.176	.200	.283	.400	.106	-.183	-.178
19	.278	.386	.404	.423	.440	.496	.616	.594	.288	.063
20	.655	.640	.650	.661	.670	.704	.773	.792	.758	.555
21	.922	.882	.885	.889	.892	.903	.926	.931	.939	.963

$$
\begin{aligned}
E_a^H[x,y] &\stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x, \text{ player hard } y, \text{ action } a\} \\
E_a^S[x,y] &\stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x, \text{ player soft } y, \text{ action } a\} \\
E_{sh}^H[x,y] &\stackrel{\text{def}}{=} \max E_s[x,y], E_h[x,y] \\
E_{sh}^H[x,y] &\stackrel{\text{def}}{=} \max E_s[x,y], E_h[x,y] \\
E_h^S[x,y] &= \sum_{1 \le k \le 10} p(k) E_{sh}^S[x, y+k] \\
E_h^H[x,y] &= p(1) E_{sh}^S[x, y+1] + \sum_{2 \le k \le 10} p(k) E_{sh}^H[x, y+k]
\end{aligned}
$$

$$E_a^H[x,y] \stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x,\text{ player hard } y,\text{ action } a\}$$

$$E_a^S[x,y] \stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x,\text{ player soft } y,\text{ action } a\}$$

after doubling down, you must take exactly one card

$$E_d^H[x,y] = p(1)E_s^S[x,y+1] + \sum_{2\le k\le 10} p(k)E_s^H[x,y+k]$$

$$E_d^H[x,y] = p(1)E_s^S[x,y+1] + \sum_{2\le k\le 10} p(k)E_s^H[x,y+k]$$

$$E_a^H[x,y] \stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x, \text{ player hard } y, \text{ action } a\}$$

$$E_a^S[x,y] \stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dealer } x, \text{ player soft } y, \text{ action } a\}$$

You must take exactly one card on each split ace

$$E_p[x,2] = 2 \cdot \sum_{1 \le k \le 10} E_s^S[x, x+k]$$

Resplitting other cards is possible

$$E_p[x,y] = 2 \cdot (p(y/2)E_p[x,y] + p(1)E_{hsd}^S[x, y/2+1] + \sum_{\substack{k \le 2 \le 10 \\ k \ne y/2}} E_{hsd}^H[x, y/2+k])$$

$$E_p[x,y] = \frac{2 \cdot (p(1)E_{hsd}^S[x, y/2+1] + \sum_{\substack{k \le 2 \le 10 \\ k \ne y/2}} E_{hsd}^H[x, y/2+k])}{1 - 2p(y/2)}$$

but $E_p[x,y]$ value could be infinite...

$$
\begin{aligned}
E[x, y] &\stackrel{\text{def}}{=} \mathbf{E}\{\text{winnings} \mid \text{dlr } x, \text{ plr } y, \text{ dlr has no blackjack}\} \\
E[10, y] &= (1 - p(1))E_{\text{legal}}[10, y] - p(1) \\
E[1, y] &= (1 - p(10))E_{\text{legal}}[1, y] - p(10) \\
E[x, y] &= (1 - p(10))E_{\text{legal}}[1, y] - p(10)
\end{aligned}
$$

Player's blackjack plays 3 : 2, unless dealer also has blackjack

$$
\begin{aligned}
E[10, bj] &= 3(1 - p(1))/2 \\
E[1, bj] &= 3(1 - p(10))/2 \\
E[x, bj] &= 3/2
\end{aligned}
$$

HIGH-LOW COUNT SYSTEMS

- Count (Tens + Aces − 2s − 3s − 4s − 5s − 6s)
- Balanced count system
- Count per deck affects both play and wager strategy
- Wagers vary from 1 unit to 10 units
- Dubner 1963, Wong 1994
- Wong predicts expected winnings of $.020 \pm .002$ units per hand

$$\begin{aligned} S &= \text{Wong's strategy} \\ B &= \textit{semi-perfect}\text{ strategy} \end{aligned}$$

SIMULATION RESULTS

Hands Simulated	Average Actual Winnings		Hands	Assessed Expectation	
	Run 1	Run 2		Run 1	Run 2
10,000	−.06128	.05743	100	.11359	−.00217
20,000	−.00184	.05600	200	.04827	−.00914
50,000	.04133	.02950	500	.02646	.01560
100,000	.03622	.01845	1,000	.01893	.00679
200,000	.02756	.03175	2,000	.01805	.01604
500,000	.02317	.02516	5,000	.01090	.01263
1,000,000	.02221	.02427	10,000	.01258	.01833
2,000,000	.02055	.02262	20,000	.01316	.01511
5,000,000	.02197	.02059	50,000	.01403	.01458
10,000,000	.01873	.02002	100,000	.01592	.01415
Variance	20			0.007	

Difference of about .04 due to approximations of EV's

UNRESOLVED ISSUES

- Assessing risk: $\mathbf{E}\{\log(\$)\}$
- Why not change the distribution to assess?
- Playing to affect the deck
- Real casino: deck isn't random, multiple players, multiple hands, etc.