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On the Order of Games

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Calistrate, Paulhus, Fraser, Hickerson, Hirshberg, Wolfe

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OUTLINE

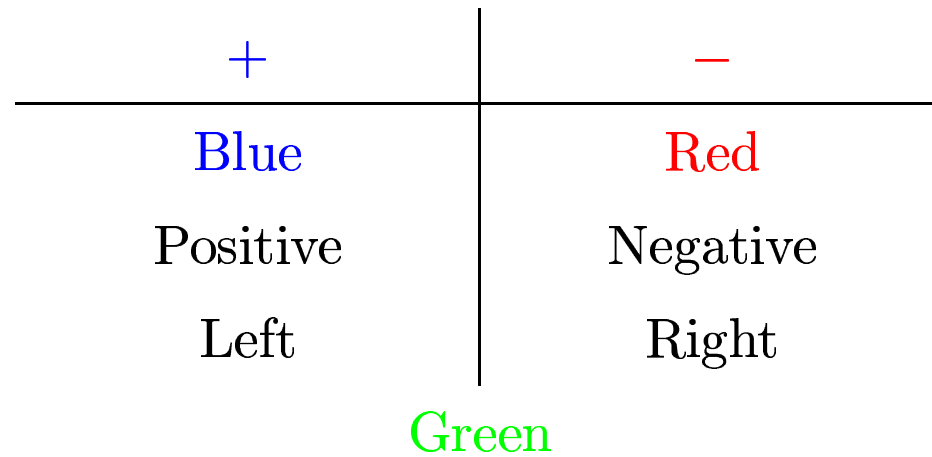
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- Hackenbush
- Game axioms
- Distributive lattices
- Lattice structure of games

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INTRODUCING LEFT AND RIGHT

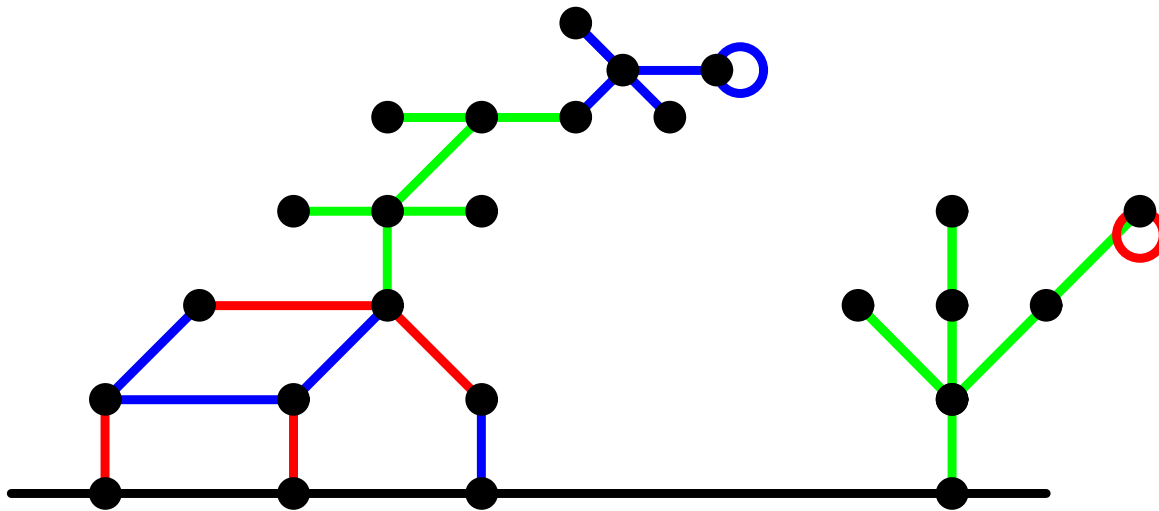
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HACKENBUSH

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GAME AXIOMS

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$$G = \{G^L \mid G^R\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \bullet \\ | \\ \text{G} \\ | \\ \bullet \end{array} = \{ \{-, \text{G} \} \mid \{-\} \}$$

$$= \{-, \text{G} \mid -\}$$

$$= -, \text{G} \mid -$$

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GAME AXIOMS

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$$G = \{G^L \mid G^R\}$$

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \bullet \\ | \\ \text{G} \\ | \\ \text{---} \end{array} = \{-, \begin{array}{c} \bullet \\ | \\ \text{G} \\ | \\ \text{---} \end{array} \mid -\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array} = \{ \mid -\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \bullet \\ | \\ \text{G} \\ | \\ \text{---} \end{array} + \begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array} = \{- + \begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array}, \begin{array}{c} \bullet \\ | \\ \text{G} \\ | \\ \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array} \mid - + \begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array}, \begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \bullet \\ | \\ \text{G} \\ | \\ \text{---} \end{array} + -\}$$

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GAME AXIOMS

-

$$G = \{G^L \mid G^R\}$$

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

$$-G = \{-G^R \mid -G^L\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{G} \\ | \\ \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array} = \{-, \begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array} \mid \begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array}\}$$

$$\begin{array}{c} \bullet \\ | \\ \text{G} \\ | \\ \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array} = \{-\begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array} \mid -\text{---}, -\begin{array}{c} \bullet \\ | \\ \text{B} \\ | \\ \text{---} \end{array}\}$$

$$= \{\begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array} \mid -, \begin{array}{c} \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array}\}$$

$$= \begin{array}{c} \bullet \\ | \\ \text{G} \\ | \\ \bullet \\ | \\ \text{R} \\ | \\ \text{---} \end{array}$$

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GAME AXIOMS

-

$$G = \{G^L \mid G^R\}$$

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

$$-G = \{-G^R \mid -G^L\}$$

$$G \geq 0 \text{ unless some } G^R \leq 0$$

“Left wins moving second on G ”

$$G \leq 0 \text{ unless some } G^L \geq 0$$

“Right wins moving second on G ”

$$G \geq H \text{ iff } G - H \geq 0$$

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GAMES BORN BY DAY n

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Define $\mathcal{G}[n]$, the games born by day n , recursively:

$$\mathcal{G}[0] \stackrel{\text{def}}{=} \{0\}$$

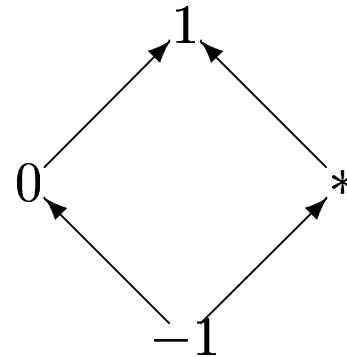
$$\mathcal{G}[n] \stackrel{\text{def}}{=} \{\{G^L \mid G^R\} : G^L, G^R \subseteq \mathcal{G}[n-1]\}$$

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GAMES BORN BY DAY 1

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		Right	
		0	\emptyset
Left	0	*	1
	\emptyset	-1	0



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GAMES BORN BY DAY 2

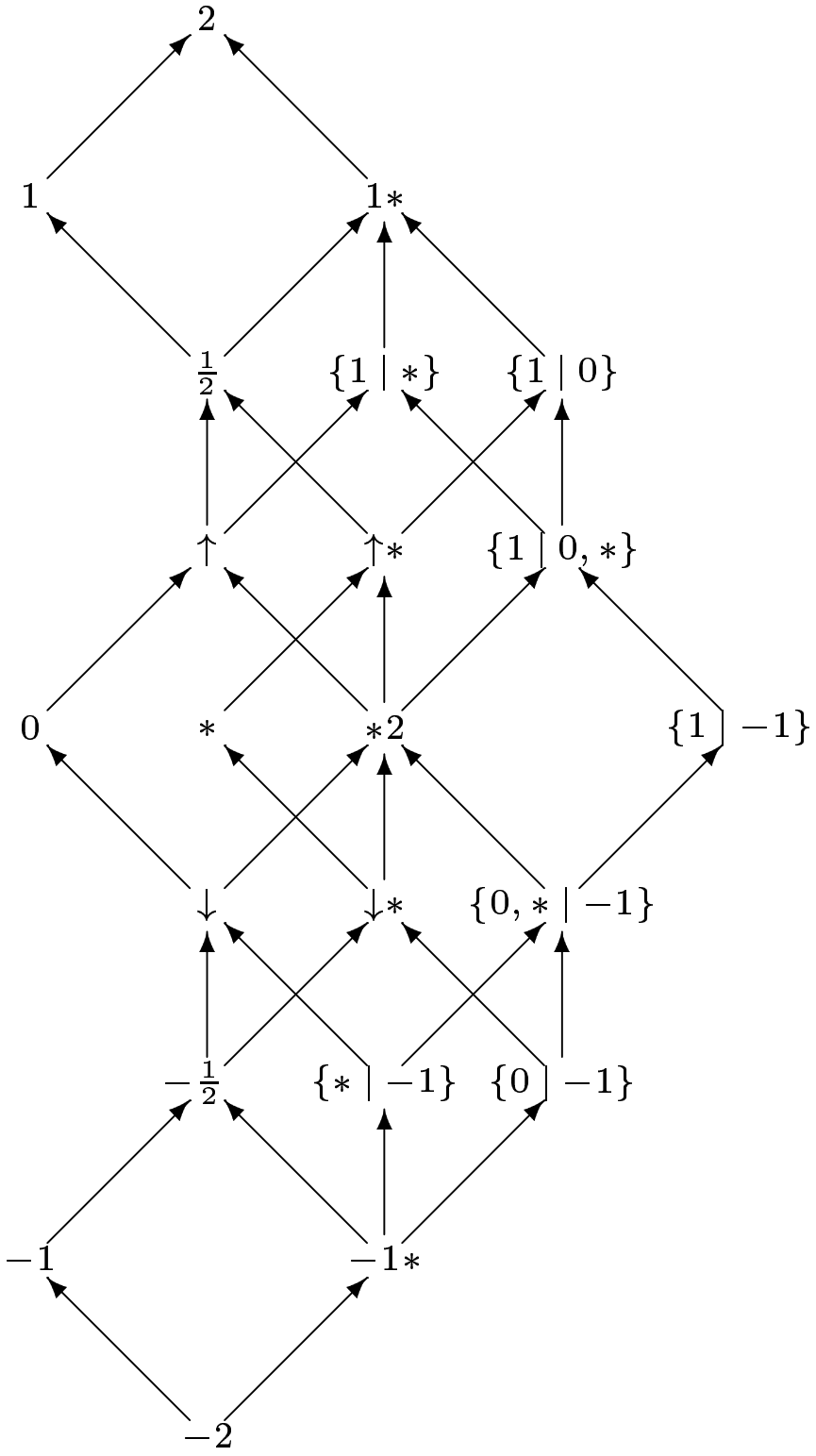
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		Right					
		-1	0, *	0	*	1	\emptyset
Left	1	± 1	1 0, *	1 0	1 *	1*	2
	0, *	0, * -1	*2	\uparrow *	\uparrow	$\frac{1}{2}$	1
	0	0 -1	\downarrow *	*			
	*	* -1	\downarrow		0		
	-1	-1*	$-\frac{1}{2}$				
	\emptyset	-2	-1				

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GAMES BORN BY DAY 2

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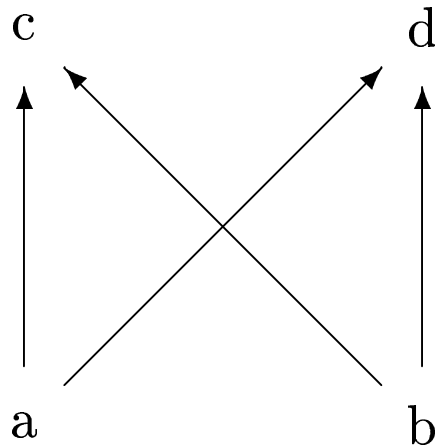
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LATTICES

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A *lattice* is a partial order in which every pair of elements a and b has

- a least upper bound or a “join” denoted $a \vee b$, and
- a greatest lower bound or “meet” denoted $a \wedge b$.



NOT A LATTICE

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DAY n LATTICE

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Theorem 1 *The games born by day n form a lattice.*

$$\lceil G \rceil \stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] : H \not\leq G\}$$

$$\lfloor G \rfloor \stackrel{\text{def}}{=} \{H \in \mathcal{G}[n-1] : H \not\geq G\}$$

$$G_1 \vee G_2 \stackrel{\text{def}}{=} \{G_1^L, G_2^L \mid \lceil G_1 \rceil \cap \lceil G_2 \rceil\}, \text{ and}$$

$$G_1 \wedge G_2 \stackrel{\text{def}}{=} \{\lfloor G_1 \rfloor \cap \lfloor G_2 \rfloor \mid G_1^R, G_2^R\}.$$

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PROOF OF DAY n LATTICE

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To show:

$$G_1 \vee G_2 \geq G_i \text{ (for } i = 1, 2), \text{ and} \quad (1)$$

$$\text{if } G \geq G_1 \text{ and } G \geq G_2 \text{ then } G \geq G_1 \vee G_2. \quad (2)$$

(1) Left wins the following game moving second:

$$(G_1 \vee G_2) - G_i$$

(2) If $G \geq G_1$ and $G \geq G_2$, then Left wins moving second on:

$$G - (G_1 \vee G_2)$$

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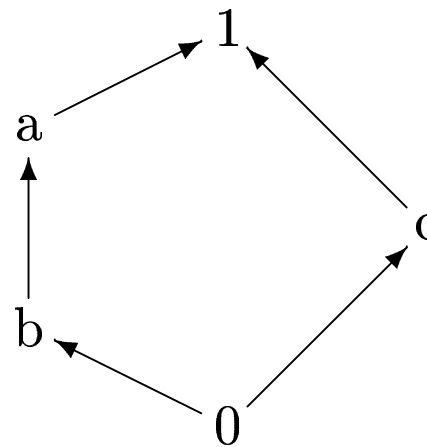
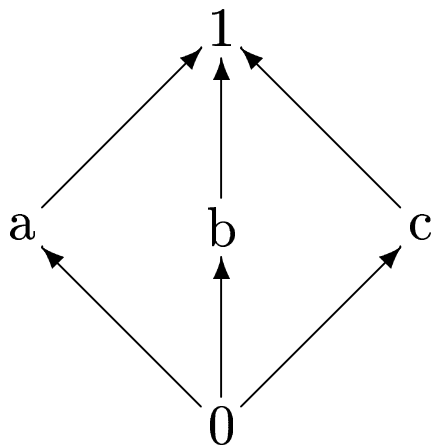
DISTRIBUTIVE LATTICES

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In a *distributive lattice*, each operation distributes over the other:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$



“THE” TWO LATTICES THAT ARE NOT DISTRIBUTIVE

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DAY n IS DISTRIBUTIVE

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Theorem 2 *The lattice of games born by day n is distributive.*

Observe,

$$[G_1 \vee G_2] = [G_1] \cup [G_2] \quad (3)$$

$$[G_1 \wedge G_2] = [G_1] \cup [G_2] \quad (4)$$

To show, $S_1 = S_2$, where

$$S_1 = H \wedge (G_1 \vee G_2)$$

$$S_2 = (H \wedge G_1) \vee (H \wedge G_2)$$

$$S_1 = \{ [H] \cap [G_1 \vee G_2] \mid H^R, [G_1] \cap [G_2] \}$$

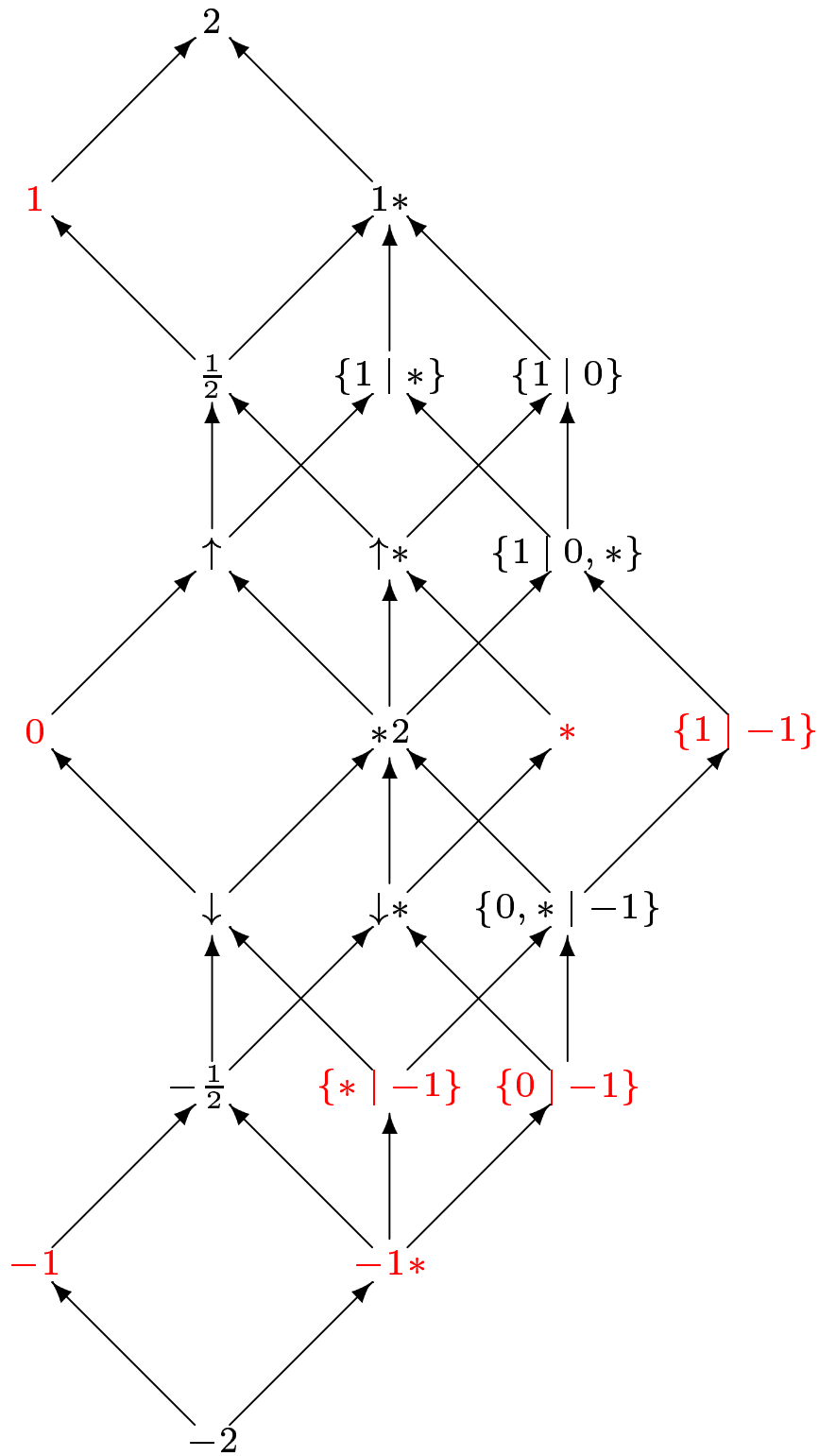
$$S_2 = \{ [H] \cap [G_1], [H] \cap [G_2] \mid [H \wedge G_1] \cap [H \wedge G_2] \}$$

$$= \{ [H] \cap [G_1 \vee G_2] \mid [H], [G_1] \cap [G_2] \}$$

+ BIRKHOFF'S REPRESENTATION THEOREM —

Partial orders are in 1-1 correspondence with distributive lattices!

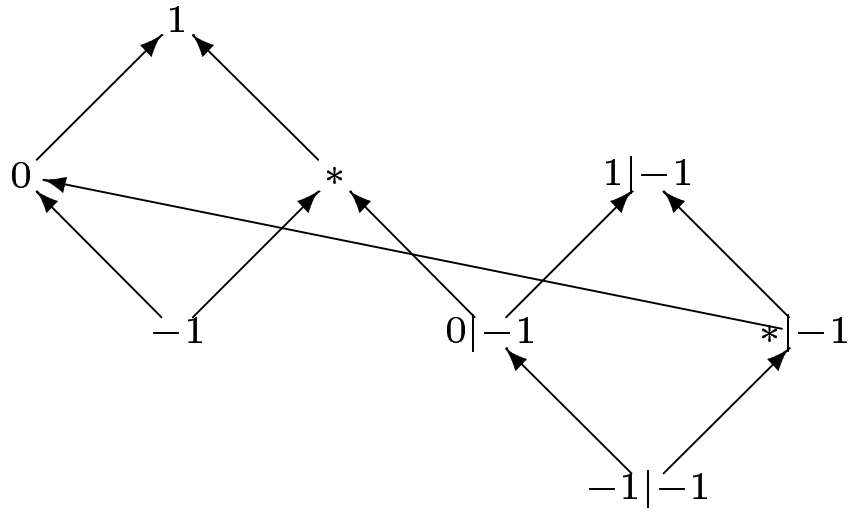
+ JOIN-IRREDUCIBLE ELEMENTS -



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P.O. OF JOIN-IRREDUCIBLES

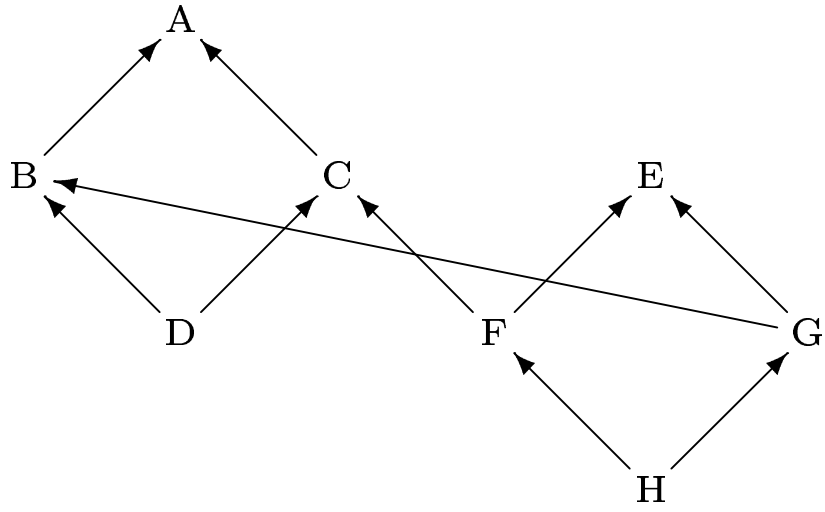
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PARTIAL ORDER

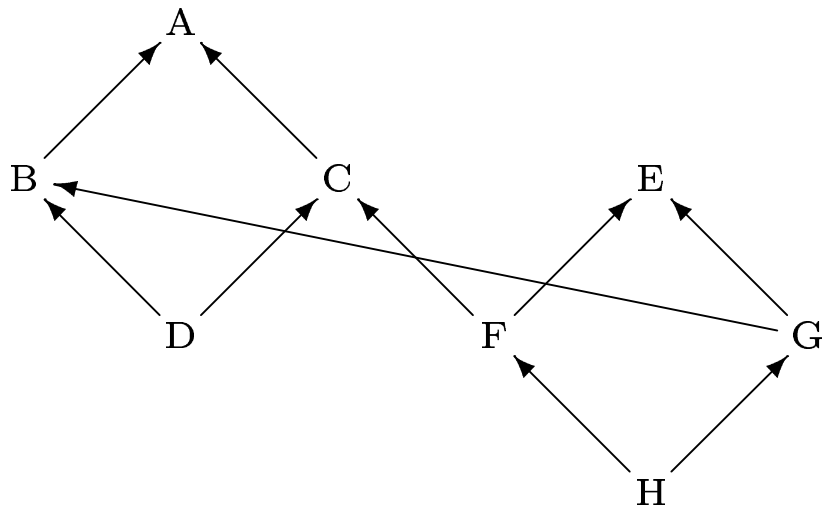
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DOWNSETS

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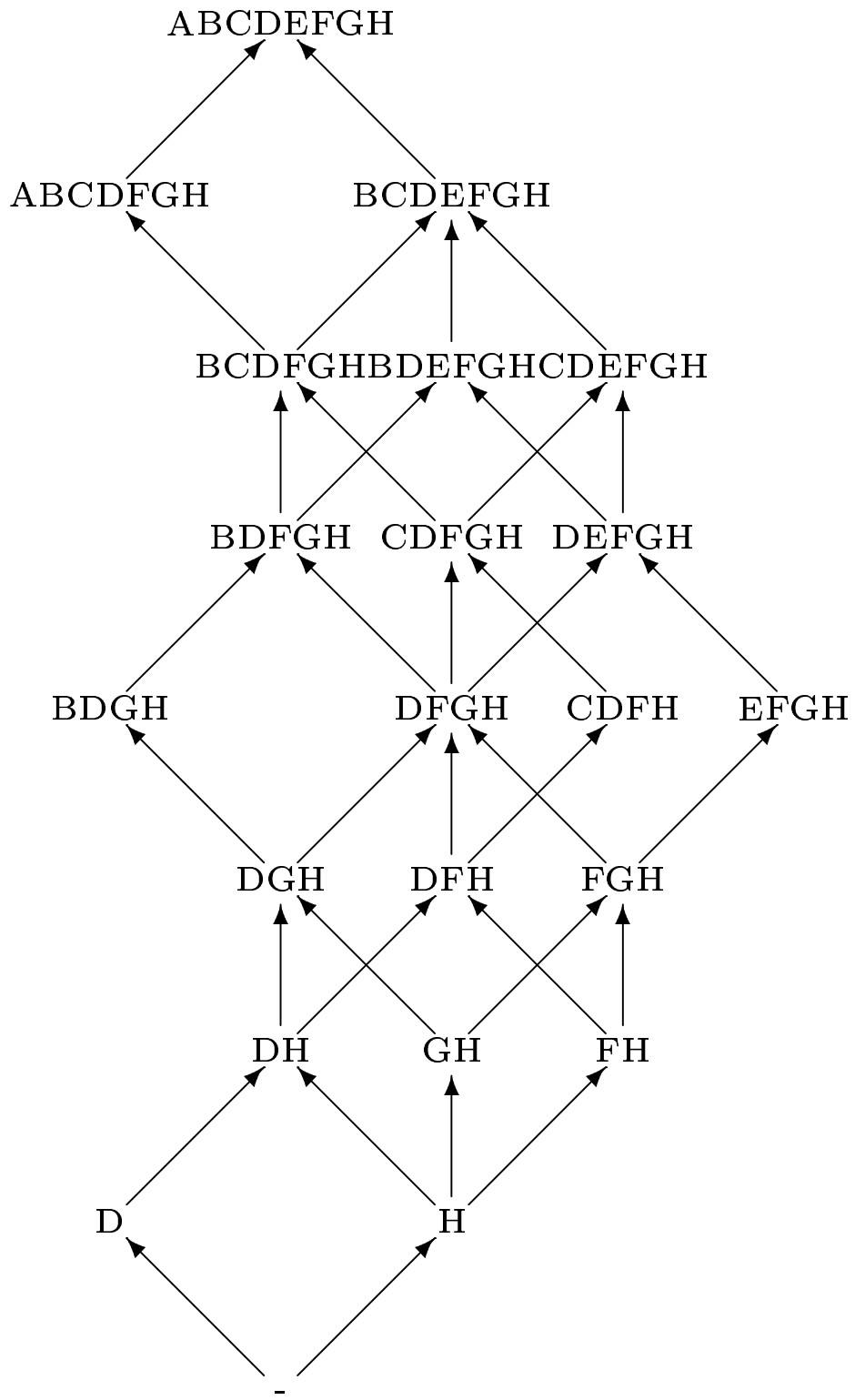


- | | |
|----------|------|
| ABCDEFGH | DFGH |
| ABCDFGH | EFGH |
| BCDEFGH | DFH |
| BCDFGH | DGH |
| BDEFGH | FGH |
| CDEFGH | DH |
| BDFGH | FH |
| CDFGH | GH |
| DEFGH | D |
| BDGH | H |
| CDFH | - |

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LATTICE OF SETS

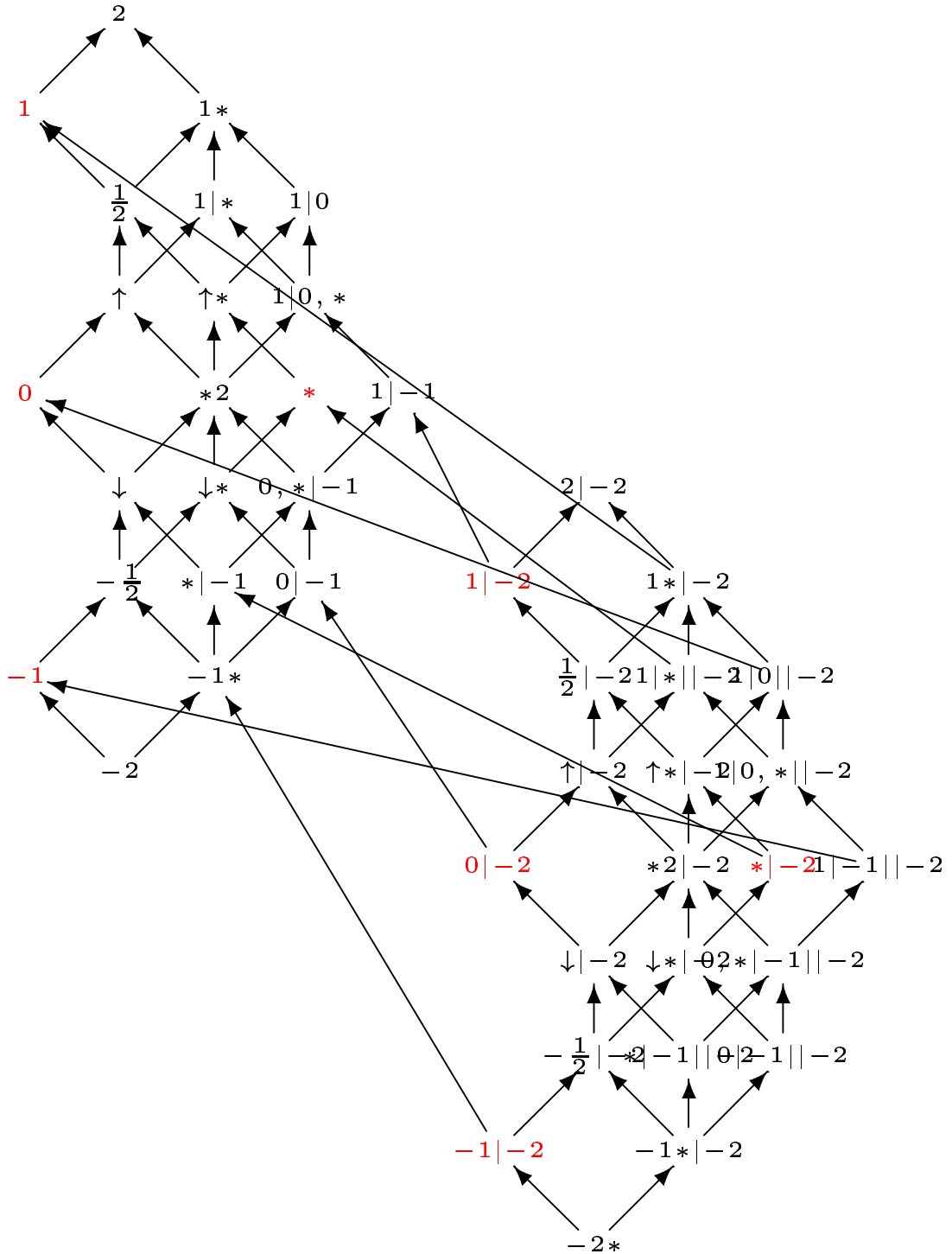
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DAY 3 JOIN-IRREDUCIBLES

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+ NUMBER OF DAY- n GAMES -

$g(n)$ $\stackrel{\text{def}}{=}$ the number of games born on day n

$$g(n + 1) \leq g(n) + 2^{g(n)} + 2$$

For all $\alpha < 1$, for sufficiently large n ,

$$g(n + 1) \geq 2^{g(n)^\alpha}$$

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ALL FINITE GAMES

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Theorem 3 *The collection of finite games, $\mathcal{G} = \bigcup_{n \geq 0} \mathcal{G}[n]$, is not a lattice.*

In fact, no two incomparable games G_1 and G_2 have a join. Define

$$H_n \stackrel{\text{def}}{=} \{G_1, G_2 \parallel G_1, G_2 \mid -n\}$$

- If $G > G_1$ and $G > G_2$ then $G > H_n$ for some n .
- $H_0 > H_1 > H_2 > \dots$