Counting the number of games

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Abstract

We give upper and lower bounds on $g(n)$ equal to the number of games born by day n. In particular, we give an upper bound of $g(n+1) \leq g(n) + 2^{g(n)} + 2$. For the lower bound, for all $\alpha < 1$, for sufficiently large n, $g(n+1) \geq 2^{g(n)^\alpha}$.

1 Introduction

For a complete introduction to combinatorial game theory, see [BCG01] or [Con01]. For a terse introduction to combinatorial game theory axioms sufficient for reading this paper, see [FHW03].

Define $\mathcal{G}_n$, the *games born by day n*, recursively as follows:

$$\begin{aligned} \mathcal{G}_0 &\stackrel{\text{def}}{=} \{0\} \\ \mathcal{G}_n &\stackrel{\text{def}}{=} \{\{\mathcal{G}^L \mid \mathcal{G}^R\} : \mathcal{G}^L, \mathcal{G}^R \subseteq \mathcal{G}_{n-1}\}. \end{aligned}$$

Previously known upper and lower bounds on the number of games, $g(n)$, born by day n are, to the best of our knowledge, unpublished. Clearly, $g(n) \leq 4^{g(n-1)}$ since there are $2^{g(n-1)}$ choices for subset G^L and for G^R. Lower bounds can be obtained by counting only those games with names. For instance, it's not hard to see that there are $2^{n+1} - 1$ *numbers* born by day n.

2 Upper bounds

Let $\mathcal{N}_n$ be the set of new games born on day $n+1$, i.e.,

$$\mathcal{N}_n = \mathcal{G}_{n+1} \setminus \mathcal{G}_n.$$

For any game $G \in \mathcal{N}_n$, define the *top cover*, $\lceil G \rceil$, the set of minimal games in $\mathcal{G}_n$ greater than G. Similarly the *bottom cover*, $\lfloor G \rfloor$, contains the maximal games in $\mathcal{G}_n$ less than G. I.e.,

$$\lceil G \rceil = \{H \in \mathcal{G}_n : H > G \text{ and for no } H' \text{ in } \mathcal{G}_n \text{ is } H > H' > G\}$$
$$\lfloor G \rfloor = \{H \in \mathcal{G}_n : H < G \text{ and for no } H' \text{ in } \mathcal{G}_n \text{ is } H < H' < G\}$$

In this paper, when a relation is applied to a game and a set it is assumed to hold for all elements of the set. We compare two sets of games similarly. For example, if S_1 and S_2 are sets, $\mathcal{S}_1 \leq \mathcal{S}_2$ if and only if for all $G_1 \in \mathcal{S}_1$ and $G_2 \in \mathcal{S}_2$, $G_1 \leq G_2$. An *anti-chain* (in the partial order $\mathcal{G}_n$) is a subset of $\mathcal{G}_n$ containing no two comparable elements. Call a pair of anti-chains, $(\mathcal{T}, \mathcal{B})$, *admissible* if $\mathcal{T} > \mathcal{B}$. In this paper, we use the symbol $G_1 \lhd\!\!\mid G_2$ to mean G_1 is less than or incomparable with G_2, i.e., $G_1 \not\geq G_2$. Similarly $G_1 \mid\!\!\rhd G_2$ if and only if $G_1 \not\leq G_2$.

Theorem 1 *There is a 1-1 correspondence between $G \in \mathcal{N}_n$ and admissible pairs $(\mathcal{T}, \mathcal{B})$. In particular, $\lceil G \rceil = \mathcal{T}$ and $\lfloor G \rfloor = \mathcal{B}$ if and only if*

$$\begin{aligned} G &= \{\mathcal{L} \mid \mathcal{R}\}, \text{ where} && (1) \\ \mathcal{L} &= \{H^L \in \mathcal{G}_n : \ H^L \lhd\!\!\mid \mathcal{T}\}, \text{ and} && (2) \\ \mathcal{R} &= \{H^R \in \mathcal{G}_n : \ H^R \mid\!\!\rhd \mathcal{B}\}, && (3) \end{aligned}$$

Proof: Note that for any G, $(\lceil G \rceil, \lfloor G \rfloor)$ is an admissible pair. The following two lemmas complete the proof. ∎

Lemma 1 *For any admissible pair $(\mathcal{T}, \mathcal{B})$, there is at most one game $G \in \mathcal{N}_n$ such that $\mathcal{T} = \lceil G \rceil$ and $\mathcal{B} = \lfloor G \rfloor$.*

Proof: Suppose one such G exists. It suffices to show $G = \{\mathcal{L} \mid \mathcal{R}\}$ where $\mathcal{L}$ and $\mathcal{R}$ are given by Equations (2) and (3). Every left option G^L is in $\mathcal{L}$ since otherwise $G^L \geq T$ for some $T \in \mathcal{T}$, and $G^L \geq G$ which is never true. Similarly each $G^R \in \mathcal{R}$. It remains to show the additional left options in $\mathcal{L}$ (and, by a parallel argument, in $\mathcal{R}$) are of no consequence. The *Gift Horse Principle* [BCG01] states that the value of game G is unaffected by introducing new left options less than or incomparable with G. But if some $H^L \in \mathcal{L}$ exceeded G then H^L (or some element between H^L and G) must be in $\mathcal{T}$. (No H^L equals G since G is a *new* day $n+1$ game.) ∎

Lemma 2 *For any admissible pair $(\mathcal{T}, \mathcal{B})$, let G be given by (1). Then $\lceil G \rceil = \mathcal{T}$ and $\lfloor G \rfloor = \mathcal{B}$.*

Proof: We'll show $\lceil G \rceil = \mathcal{T}$. (The case $\lfloor G \rfloor = \mathcal{B}$ is symmetric.)

We'll first show that if $T \in \mathcal{T}$ then $T > G$ by exhibiting a winning strategy for Left (moving first or second) on $T - G$. Since $T > \mathcal{B}$, $T \in \mathcal{R}$ and Left can win moving first to $T - T$. If Right moves first to some $T - H^L$ for $H^L \in \mathcal{L}$, Left has a winning response since $T \mid\!\!\rhd H^L$. Lastly, if Right moves on T to some $T^R - \{\mathcal{L} \mid \mathcal{R}\}$, observe that $T^R \mid\!\!\rhd T > \mathcal{B}$, and hence $T^R \in \mathcal{R}$ and Left plays to $T^R - T^R$.

Next, we'll prove that if $T' \in \mathcal{G}_n$ and $T' \geq G$ then $T' \geq T$ for some $T \in \mathcal{T}$, establishing the lemma. Suppose, to the contrary, that $T' \lhd\!\!\mid \mathcal{T}$. Then $T' \in \mathcal{L}$

and Right wins moving first from $T' - G$ to $T' - T'$ and so $T' \lhd\!\mid G$. ■

Corollary 1.1 *(to Theorem 1) For any subset $\mathcal{S}$ of $\mathcal{G}_n$, define*

$$f(\mathcal{S}) = |\{G \in \mathcal{N}_n : \ \mathcal{S} = \lceil G \rceil \cup \lfloor G \rfloor\}|.$$

Then $f(\mathcal{S}) \leq 2$. In particular,

1. *$f(\mathcal{S}) = 1$ if and only if $\mathcal{S}$ is the union of non-empty anti-chains $\mathcal{T} \cup \mathcal{B}$ with $\mathcal{T} > \mathcal{B}$, and*
2. *$f(\mathcal{S}) = 2$ if and only if $\mathcal{S}$ is an anti-chain.*
3. *In all other cases, $f(\mathcal{S}) = 0$.*

We need only use $f(\mathcal{S}) \leq 2$ to show $|\mathcal{N}_n|$ is bounded by twice the number of subsets of day n games, proving the following theorem of Dean Hickerson's [Hic02]:

Theorem 2 $|\mathcal{N}_n| \leq 2^{g(n)+1}$

Dan Hoey [Hoe02] tightened this upper bound by using Corollary 1.1 more strongly:

Theorem 3 $g(n+1) \leq g(n) + 2^{g(n)} + 2$.

Proof: On day 0, the theorem holds. On subsequent days, the partial order of $\mathcal{G}_n$ has a top and bottom (n and $-n$) each comparable to all other elements in $\mathcal{G}_n$. Hence, no subset $\mathcal{S}$ of $\mathcal{G}_n$ containing n or $-n$ will have an isolated element (incomparable with all other games in $\mathcal{S}$) unless $\mathcal{S}$ is the singleton set $\{n\}$ or $\{-n\}$, and any subset $\mathcal{S}$ of $\mathcal{G}_n$ containing both n and $-n$ will have a 3-chain unless $\mathcal{S} = \{n, -n\}$. So,

$$\begin{aligned}
|\mathcal{N}_n| &\leq |\{\mathcal{S} \subseteq \mathcal{G}_n : \mathcal{S} \text{ has no 3-chain and at least one isolated element}\}| \\
&\quad + |\{\mathcal{S} \subseteq \mathcal{G}_n : \mathcal{S} \text{ has no 3-chain}\}| \\
&\leq (2 + 2^{g(n)-2} - 1) + (4 + 3(2^{g(n)-2} - 1)) \\
&= 2 + 2^{g(n)}
\end{aligned}$$

■

This bound can be tightened still further by making stronger use of the fact that $\mathcal{S}$ cannot have a 3-chain. For example,

Theorem 4

$$g(n+1) \leq g(n) + [g(n-1)^2 + \frac{5}{2}g(n-1) + 2] \cdot 2^{g(n)-2g(n-1)}$$

(The right hand side is upper bounded by $[2g(n-1)^2/4^{g(n-1)}] \cdot 2^{g(n)}$ for $n \geq 2$.)

Proof: The length of the longest chain of games born by day n is exactly $2g(n-1)+1$ [FHW03]; call this value k. Then the number of possibilities for the elements of $\mathcal{S}$ in such a chain is at most $\binom{k}{2}+k+1$. When two elements are taken from the chain, $\mathcal{S}$ determines at most one game in $\mathcal{G}_{n+1}$. The number of possibilities for elements of $\mathcal{S}$ outside the chain is at most $2^{g(n)-k}$. Hence,

$$\begin{aligned} g(n+1) &\leq g(n) + \left(\binom{k}{2} + 2(k+1)\right) 2^{g(n)-k} \\ &\leq g(n) + \left[g(n-1)^2 + \frac{5}{2}g(n-1) + 2\right] \cdot 2^{g(n)-2g(n-1)} \end{aligned}$$

■

3 Lower bounds

In this section we give a lower bound of $g(n) \geq 2^{g(n-1)^\alpha}$ where $\alpha > .51$ and $\alpha \longrightarrow 1$ as $n \longrightarrow \infty$. In addition, if $a(n)$ is the longest day n anti-chain, we show $a(n+1) \geq \binom{a(n)}{\lfloor a(n)/2 \rfloor}$.

We'll first bound $g(n+1)$ in two ways: the first expression is simpler, and the second is tighter.

Theorem 5

$$g(n+1) \geq 2^{g(n)/2g(n-1)}, \textit{ and} \tag{4}$$

$$g(n+1) \geq (8g(n-1)-4)\left(2^{(g(n)-2)/(2g(n-1)-1)} - 1\right). \tag{5}$$

Proof: The games born on day n form a distributive lattice [CPW02], and the length of every maximal chain in the lattice is exactly $l = 2g(n-1)+1$ [FHW03]. To obtain the first inequality, observe that one anti-chain must be of length $\geq g(n)/l$. By Theorem 1, each non-empty anti-chain $\mathcal{S}$ determines 4 day $n+1$ games, those with admissible pairs $(\mathcal{S}, \{-n\})$, $(\mathcal{S}, \{\})$, $(\{n\}, \mathcal{S})$, and $(\{\}, \mathcal{S})$. So,

$$g(n+1) \geq 4 \cdot 2^{g(n)/(2g(n-1)+1)} - 1$$

which we bound to give (4).

We can tighten the bound by counting all single-level anti-chains. On day $n > 0$, the extreme (top and bottom) elements are $\pm n$. Using the remaining $g(n)-2$ elements, we'll bound the number of non-empty anti-chains occupying a single non-extreme level by $(g(n)-2)/(l-2)$. If these levels have $a_2, \ldots a_{l-1}$ elements, then the number of non-empty anti-chains occupying a single level is $\sum_i (2^{a_i}-1)$ which, by the convexity of 2^x, we can bound by summing the average length of an anti-chain:

$$\sum_{2 \leq i \leq l-1} (2^{a_i} - 1) = \sum_i 2^{a_i} - (l-2) \geq (l-2)\left(2^{(g(n)-2)/(l-2)} - 1\right).$$

Again, each non-empty anti-chain yields 4 games, giving (5). ∎

Lemma 3 $g(n) \geq g(n-1)^2$.

Proof: The Lemma is true for $n < 5$, for the number of games born by day n are 1, 4, 22, and 1474, for $n = 0, 1, 2$, and 3. Applying (5) yields $g(4) \geq 3 \times 10^{12}$. Otherwise, applying induction to(4),

$$g(n) \geq 2^{g(n-1)/2g(n-2)} \geq 2^{\sqrt{g(n-1)}/2} \geq g(n-1)^2.$$

In the last step, note $2^{\sqrt{x}/2} \geq x^2$ when $x \geq 2000$, i.e., $g(n-1) \geq 2000$ or $n \geq 5$. ∎

Theorem 6 $g(n) = 2^{g(n-1)^{\alpha(n)}}$ *where* $\alpha(n) > .51$ *and* $\alpha(n) \longrightarrow 1$ *as* $n \longrightarrow \infty$.

Proof: Solving for $\alpha(n)$, and writing lg to mean $\log_2$,

$$\begin{aligned}
\alpha(n) &= \frac{\lg \lg g(n)}{\lg g(n-1)} \\
&\geq \frac{\lg g(n-1) - \lg(2g(n-2))}{\lg g(n-1)} \qquad (6) \\
&= 1 - \frac{1 + \lg g(n-2)}{\lg g(n-1)} \\
&\geq 1 - \frac{1 + \lg g(n-2)}{g(n-2)/2g(n-3)} \\
&\geq 1 - \frac{1 + \lg g(n-2)}{\frac{1}{2}\sqrt{g(n-2)}} \qquad (7)
\end{aligned}$$

This last quantity monotonically increases in n for $n \geq 3$ and limits to 1. For $n \leq 3$, $\alpha(n)$ can be calculated exactly from known values. Bounding $g(4)$ by (5) yields $\alpha(4) > .51$. Using (6), $\alpha(4) > .72$. Using (7) and monotonicity, $\alpha(n) > .99995$ for $n \geq 6$. ∎

Finally, define $a(n)$ to be the length of the longest anti-chain on day n. Since $g(n+1) \geq 2^{a(n)}$, the following lower bound on $a(n)$ suggests a faster order of growth for $\{g(n)\}$ than Theorems 6 and 5.

Theorem 7

$$a(n+1) \geq \binom{a(n)+1}{\lceil a(n)/2 \rceil} \geq 2^{a(n)}/\sqrt{a(n)}$$

Proof: An upper bound of $\binom{a(n)}{\lfloor a(n)/2 \rfloor}$ uses elementary techniques. Let the longest day n anti-chain be $\mathcal{A}(n)$. The set of games

$$\{\{n \,|\, \mathcal{S}\} : \ \mathcal{S} \subset \mathcal{A}(n) \text{ and } |\mathcal{S}| = \lfloor a(n)/2 \rfloor\}$$

is an anti-chain: Left can win moving first on the difference of any pair $\{n|S_1\} - \{n|S_2\}$ by moving to $\{n|S_1\} - G$ where $G \in S_2 \setminus S_1$.

The proof of the theorem requires knowledge of results from [FHW03]. Construct $A'(n)$ from $A(n)$ with the one additional game $\{n \mid -n\}$. All games in $A'(n)$ are incomparable and join-irreducible in the day $n+1$ distributive lattice. Let $J(\mathcal{S})$ be the day $n+1$ join of elements in $\mathcal{S}$. Birkhoff's construction of the day $n+1$ lattice from the join-irreducibles guarantees that

$$\{\{J(\mathcal{S})\} : \ \mathcal{S} \subset \mathcal{A}'(n) \text{ and } |\mathcal{S}| = \lceil |a(n)|/2 \rceil\}$$

is an anti-chain. This set has size $\binom{a(n)+1}{\lceil a(n)/2 \rceil}$ which, by Sterling's approximation, is about $2^{1+a(n)}/\sqrt{a(n)\cdot\pi/2} \geq 2^{a(n)}/\sqrt{a(n)}$. ■

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