Games Played in Tall Warehouses

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(based on Michael Albert, Richard Nowakowski)

Partizan End Nim

Players: Left and Right

Position: n stacks of boxes of varying ordinal heights in a row.

Moves: Left (Right) reduces the height of the leftmost (rightmost) non-empty stack.

Winner: The player to remove the last box wins.

Partizan End Nim is all-small.

Sample game

$$3523319 \xrightarrow{L} 2523319 \xrightarrow{R} 252331 \xrightarrow{L} 52331 \xrightarrow{R} 5233 \xrightarrow{L} 4233 \xrightarrow{R} 4232 \xrightarrow{L} 3232 \xrightarrow{R} 323 \xrightarrow{L} 223 \xrightarrow{R} 22 \xrightarrow{L} 2 \xrightarrow{R} -$$

Outcomes

- $\mathcal{N}$ if the 1st (or Next) player always has a winning strategy.
- $\mathcal{P}$ if the 2nd (or Previous) player always has a winning strategy.
- $\mathcal{L}$ if the Left player always has a winning strategy.
- $\mathcal{R}$ if the Right player always has a winning strategy.

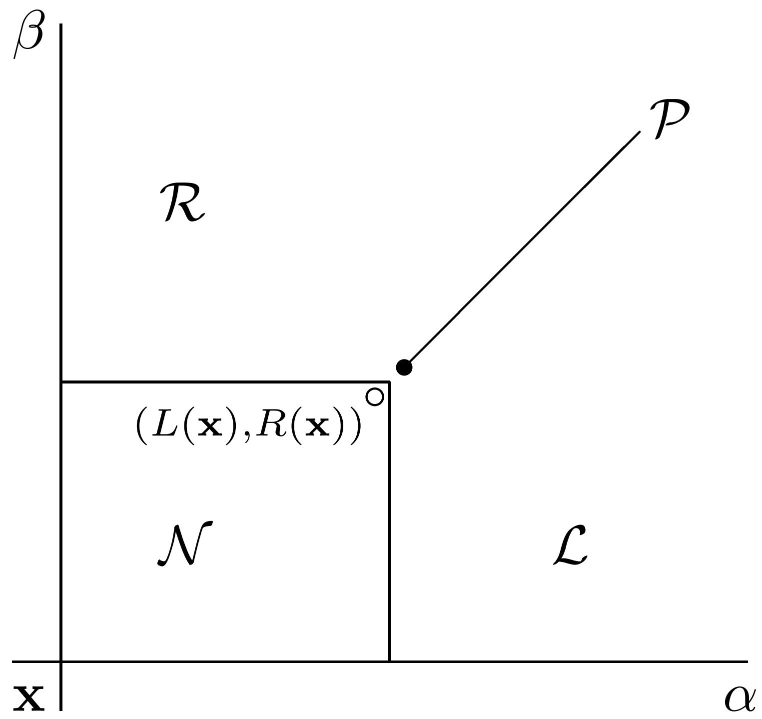
$L(\mathbf{x})$ and $R(\mathbf{x})$

$$L(\mathbf{x}) \stackrel{\text{def}}{=} \text{minimum } \alpha \geq 0 \text{ s.t. Left wins } \alpha\mathbf{x} \text{ moving second}$$

$$R(\mathbf{x}) \stackrel{\text{def}}{=} \text{minimum } \beta \geq 0 \text{ s.t. Right wins } \mathbf{x}\beta \text{ moving second}$$

$$\alpha\mathbf{x}\beta \in \begin{cases} \mathcal{L} & \text{if } \alpha > L(\mathbf{x}\beta) \text{ and } \beta \leq R(\alpha\mathbf{x}) \\ \mathcal{R} & \text{if } \alpha \leq L(\mathbf{x}\beta) \text{ and } \beta > R(\alpha\mathbf{x}) \\ \mathcal{N} & \text{if } \alpha > L(\mathbf{x}\beta) \text{ and } \beta > R(\alpha\mathbf{x}) \\ \mathcal{P} & \text{if } \alpha \leq L(\mathbf{x}\beta) \text{ and } \beta \leq R(\alpha\mathbf{x}) \end{cases}$$

Triple point



Computing $L(\mathbf{x})$ and $R(\mathbf{x})$

$$
\begin{aligned}
R(\alpha\mathbf{x}) &= \begin{cases} 0 & \text{if } \alpha \leq L(\mathbf{x}) \\ R(\mathbf{x}) + (\alpha - L(\mathbf{x})) & \text{if } \alpha > L(\mathbf{x}) \end{cases} \\
L(\mathbf{x}\beta) &= \begin{cases} 0 & \text{if } \beta \leq R(\mathbf{x}) \\ L(\mathbf{x}) + (\beta - R(\mathbf{x})) & \text{if } \beta > R(\mathbf{x}) \end{cases}
\end{aligned}
$$

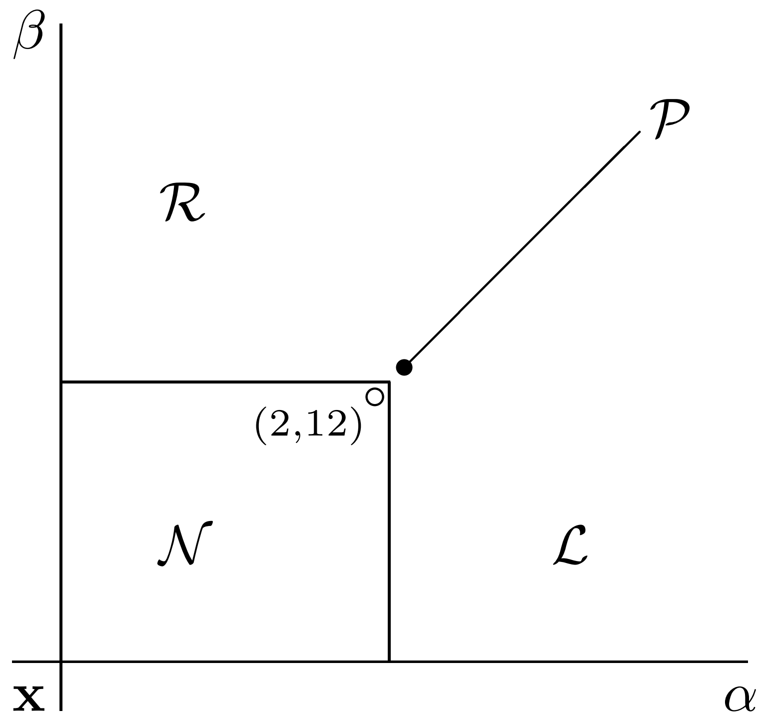
Example: 3523319

$$L(n) = R(n) = n$$

$$R(mn) = \begin{cases} 0 & \text{if } m \leq n \\ m & \text{if } m > n \end{cases} \qquad L(mn) = \begin{cases} 0 & \text{if } m \geq n \\ n & \text{if } m < n \end{cases}$$

$\mathbf{x}$	$L(\mathbf{x})$	$R(\mathbf{x})$
523	0	2
233	6	2
331	1	6
5233	1	0
2331	0	7
52331	2	12

Example: 3523319



Ordinals

$$
\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} = \{\emptyset\} \\
2 &= \{1\} = \{\emptyset, \{\emptyset\}\} \\
&\vdots \\
\omega &= \{0, 1, 2, 3, \ldots\} \\
\omega + 1 &= \{0, 1, 2, 3, \ldots, \omega\}
\end{aligned}
$$

Ordinal addition and subtraction

$$\alpha + \beta \stackrel{\text{def}}{=} \alpha \cup \{\alpha + \beta'\}_{\beta' \in \beta}$$
$$\alpha - \beta \stackrel{\text{def}}{=} \{\alpha' - \beta\}_{\alpha' \geq \beta, \alpha' \in \alpha}$$

(Add to the top, subtract from the bottom.)

Not commutative: $1 + \omega = \omega \neq \omega + 1$.

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$
$$(\beta + \alpha) - \beta = \alpha$$
$$\beta + (\alpha - \beta) = \alpha$$

Atomic weights: $x = 132$

7	-4	-4	-4	-4	-4	-4	-4	P
6	-4	-4	-4	-4	-4	-4	P	4
5	-4	-4	-4	-4	-4	P	4	4
4	-1	-1	-1	-1	0	4	4	4
3	0	0	0	0	1	4	4	4
2	0	0	0	0	1	4	4	4
1	0	0	0	0	1	4	4	4
	1	2	3	4	5	6	7	8

Atomic weights: $x = 233$

7	-4	-4	-4	-4	-4	-4	-4	-4
6	-4	-4	-4	-4	-4	-4	-4	-4
5	-4	-4	-4	-4	-4	-4	-4	-4
4	-4	-4	-4	-4	-4	-4	-4	P
3	-2	-2	-2	-2	-2	-2	P	4
2	-1	-1	-1	-1	-1	-1	4	4
1	-1	-1	-1	-1	-1	0	4	4
	1	2	3	4	5	6	7	8