Go Endgames are PSPACE hard

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PREVIOUS RESULTS

- PSPACE-hard
  Sipser & Lichtenstein (1978)

- EXPTIME-complete
  Robson (1982)

- Sums of games
  Morris (1981)

- Sums of simple games
  Yedwab (1985), Moews (1994)

- Go endgames
  Berlekamp & Wolfe (1991)

REDUCTIONS

Problem A **reduces to** problem B if there is an efficient (polynomial time) transformation which translates

- **yes** instances of problem A to **yes** instances of problem B, and
- **no** instances of problem A to **no** instances of problem B.

If A reduces to B, then an efficient algorithm for B yields an efficient one for A.

Intuition: B is computationally at least as hard as A.

PROOF OUTLINE

3-QBF

$\Downarrow$ Yedwab

PARTITION GAME

$\Downarrow$ Yedwab, Moews

SWITCH GAME

$\Downarrow$

FRACTIONAL SWITCH GAME

$\Downarrow$ (Yedwab, Moews)

GAME SUM

$\Downarrow$

GO ENDGAME

PARTITION GAME

$X_1 = 3$	$\overline{X_1} = 4$
$X_2 = 2$	$\overline{X_2} = 7$
$X_3 = 1$	$\overline{X_3} = 4$
$X_4 = 0$	$\overline{X_4} = 0$
$X_5 = 1$	$\overline{X_5} = 5$

$$S = 14$$

L chooses X_1 or $\overline{X_1}$
R chooses X_2 or $\overline{X_2}$
L chooses X_3 or $\overline{X_3}$
R chooses X_4 or $\overline{X_4}$
⋮

Left wins if the chosen numbers

sum to S

$$\exists x_1 \forall x_2 \exists x_3 : (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3)$$

	x_1	$\overline{x_1}$	x_2	$\overline{x_2}$	x_3	x_3	C_1	C_2
$X_1 =$	1	0	0	0	0	0	0	0
$\overline{X_1} =$	0	1	0	0	0	0	0	0
$X_2 =$	0	0	1	0	0	0	0	0
$\overline{X_2} =$	0	0	0	1	0	0	0	0
$X_3 =$	0	0	0	0	1	1	0	0
$\overline{X_3} =$	0	0	0	0	0	0	0	0
$X_5 =$	1	0	0	0	0	0	1	0
$X_7 =$	0	0	1	0	0	0	1	0
$X_9 =$	0	0	0	0	1	0	1	0
$X_{11} =$	0	1	0	0	0	0	0	1
$X_{13} =$	0	0	0	1	0	0	0	1
$X_{15} =$	0	0	0	0	0	1	0	1
$X_{17} =$	1	0	0	0	0	0	0	0
$\vdots$				$\vdots$				
$X_{27} =$	0	0	0	0	0	1	0	0
$S =$	1	1	1	1	1	1	1	1

Base 4. Omitted X_i and $\overline{X_i}$ are 0.

SWITCH GAMES

The *switch game*

$$\pm 15 \quad \pm 10 \quad \pm 12 \quad \pm 20 \quad \pm 6 \quad \pm 5$$

has *outcome:*

$$20 - 15 + 12 - 10 + 6 - 5 = 8$$

Note: Bypassing a highest number of odd multiplicity costs...

SWITCH GAME

$X_1 = \pm 5$	$\overline{X_1} = \pm 7$
$X_2 = \pm 4$	$\overline{X_2} = \pm 2$
$X_3 = \pm 5$	$\overline{X_3} = \pm 4$
$X_4 = \pm 6$	$\overline{X_4} = \pm 2$

$$S = 3$$

L chooses X_1 or $\overline{X_1}$
R chooses X_2 or $\overline{X_2}$
⋮

Left wins if the chosen switches

have outcome S

PARTITION GAME:

$X_1 = 3$	$\overline{X_1} = 4$
$X_2 = 2$	$\overline{X_2} = 7$
$X_3 = 1$	$\overline{X_3} = 4$
$X_4 = 0$	$\overline{X_4} = 0$
$X_5 = 1$	$\overline{X_5} = 5$

$$S = 14$$

SWITCH GAME:

$X_1 = \pm(2^{20} + 3)$	$\overline{X_1} = \pm(2^{20} + 4)$
$X_2 = \pm(2^{19} - 2)$	$\overline{X_2} = \pm(2^{19} - 7)$
$X_3 = \pm(2^{18} + 1)$	$\overline{X_3} = \pm(2^{18} + 4)$
$X_4 = \pm(2^{17} - 0)$	$\overline{X_4} = \pm(2^{17} - 0)$
$X_5 = \pm(2^{16} + 1)$	$\overline{X_5} = \pm(2^{16} + 5)$

$$S = 14 + 2^{20} - 2^{19} + 2^{18} - 2^{17} + 2^{16}$$

SWITCH GAME:

$X_1 = \pm 5$	$\overline{X_1} = \pm 7$
$X_2 = \pm 4$	$\overline{X_2} = \pm 2$
$X_3 = \pm 5$	$\overline{X_3} = \pm 4$
$X_4 = \pm 6$	$\overline{X_4} = \pm 2$

$$S = 3$$

FRACTIONAL SWITCH GAME:

$X_1 = \pm\frac{5}{8} = \pm.101$	$\overline{X_1} = \pm\frac{7}{8} = \pm.111$
$X_2 = \pm\frac{4}{8} = \pm.100$	$\overline{X_2} = \pm\frac{2}{8} = \pm.010$
$X_3 = \pm\frac{5}{8} = \pm.101$	$\overline{X_3} = \pm\frac{4}{8} = \pm.100$
$X_4 = \pm\frac{6}{8} = \pm.110$	$\overline{X_4} = \pm\frac{2}{8} = \pm.010$

$$S = .011$$

GAME SUM

$$5\tfrac{1}{4} \;\|\; 1\tfrac{3}{4} \;|\; 1$$

$$-\tfrac{5}{8} \;\|\; -4\tfrac{1}{8} \;|\; -5\tfrac{1}{8}$$

$$\tfrac{1}{8} \;\|\; -1\tfrac{1}{4} \;|\; -4\tfrac{3}{8}$$

$$0 \;|\; 0 \;\|\; 0$$

Can Left force a win, i.e., force

- a positive final score, or
- a zero score with Left moving last?

$X_1 = \pm\frac{1}{2}$	$\overline{X_1} = \pm\frac{5}{8}$
$X_2 = \pm\frac{1}{4}$	$\overline{X_2} = \pm\frac{3}{8}$
$X_3 = \pm\frac{3}{8}$	$\overline{X_3} = \pm\frac{1}{8}$
$X_4 = \pm\frac{1}{4}$	$\overline{X_4} = \pm 0$

$S = \frac{3}{8}$

$$
\begin{array}{c|c|c}
800 & -800 + 10 + \frac{1}{2} & -800 - 10 - \frac{1}{2} \\
800 & -800 + 10 + \frac{5}{8} & -800 - 10 - \frac{5}{8} \\
0 & -700 & -700
\end{array}
$$

(with $||$ after the first column)

$$
\begin{array}{c|c|c}
600 + 10 + \frac{1}{4} & 600 - 10 - \frac{1}{4} & -600 \\
600 + 10 + \frac{3}{8} & 600 - 10 - \frac{3}{8} & -600 \\
500 & 500 & 0
\end{array}
$$

(with $||$ after the second column)

$$
\begin{array}{c|c|c}
400 & -400 + 10 + \frac{3}{8} & -400 - 10 - \frac{3}{8} \\
400 & -400 + 10 + \frac{1}{8} & -400 - 10 - \frac{1}{8} \\
0 & -300 & -300
\end{array}
$$

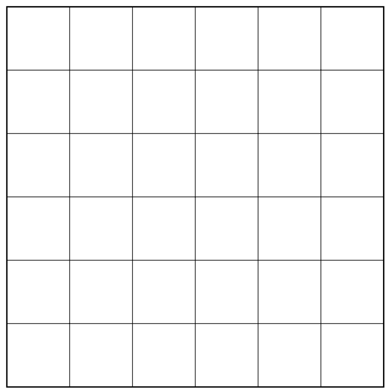
(with $||$ after the first column)

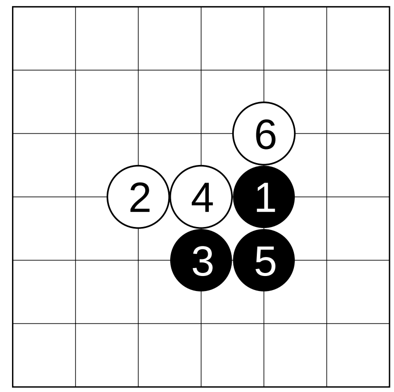
$$
\begin{array}{c|c|c}
200 + 10 + \frac{1}{4} & 200 - 10 - \frac{1}{4} & -200 \\
200 + 10 + 0 & 200 - 10 - 0 & -200
\end{array}
$$

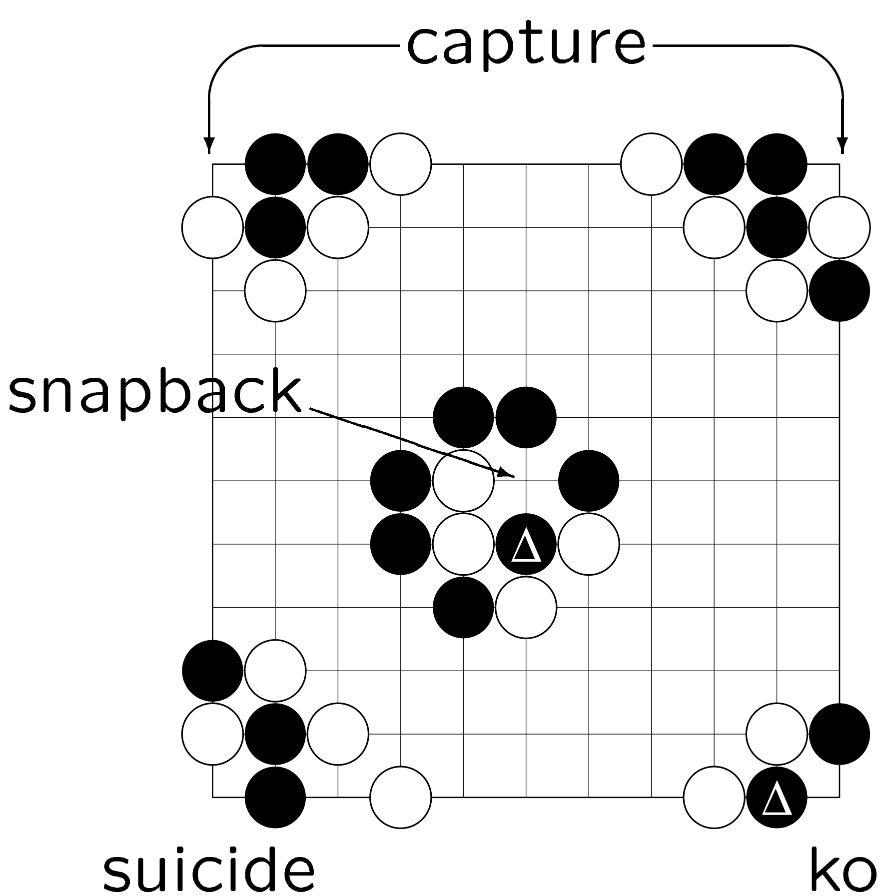
(with $||$ after the second column)

$$
\begin{array}{ccccc}
50 & | & -\frac{3}{8} & || & -100 \\
100 + \frac{3}{8} & || & 0 & | & 0
\end{array}
$$

$$0 \;||\; 0 \;|\; 0 \quad + \quad 0 \;||\; 0 \;|\; 0$$




6
2
4
1
3
5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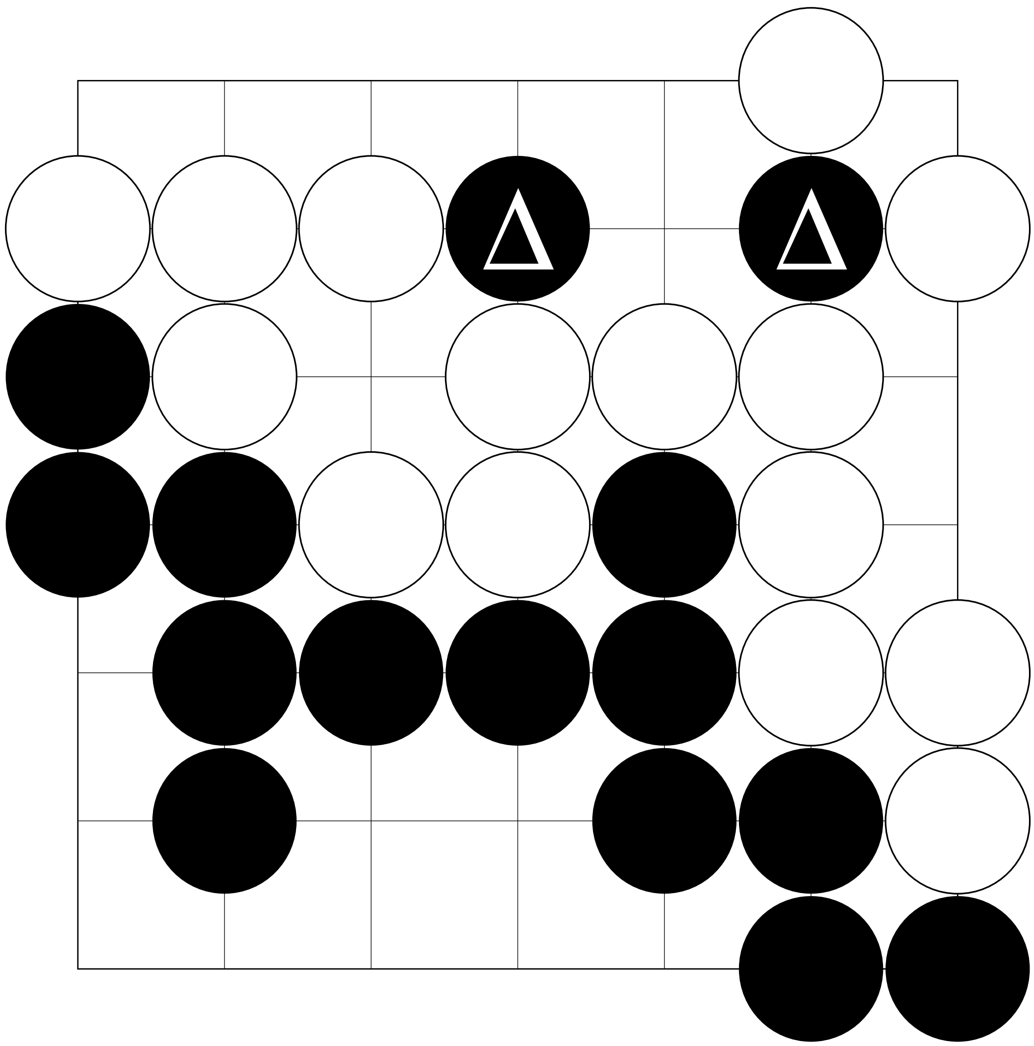
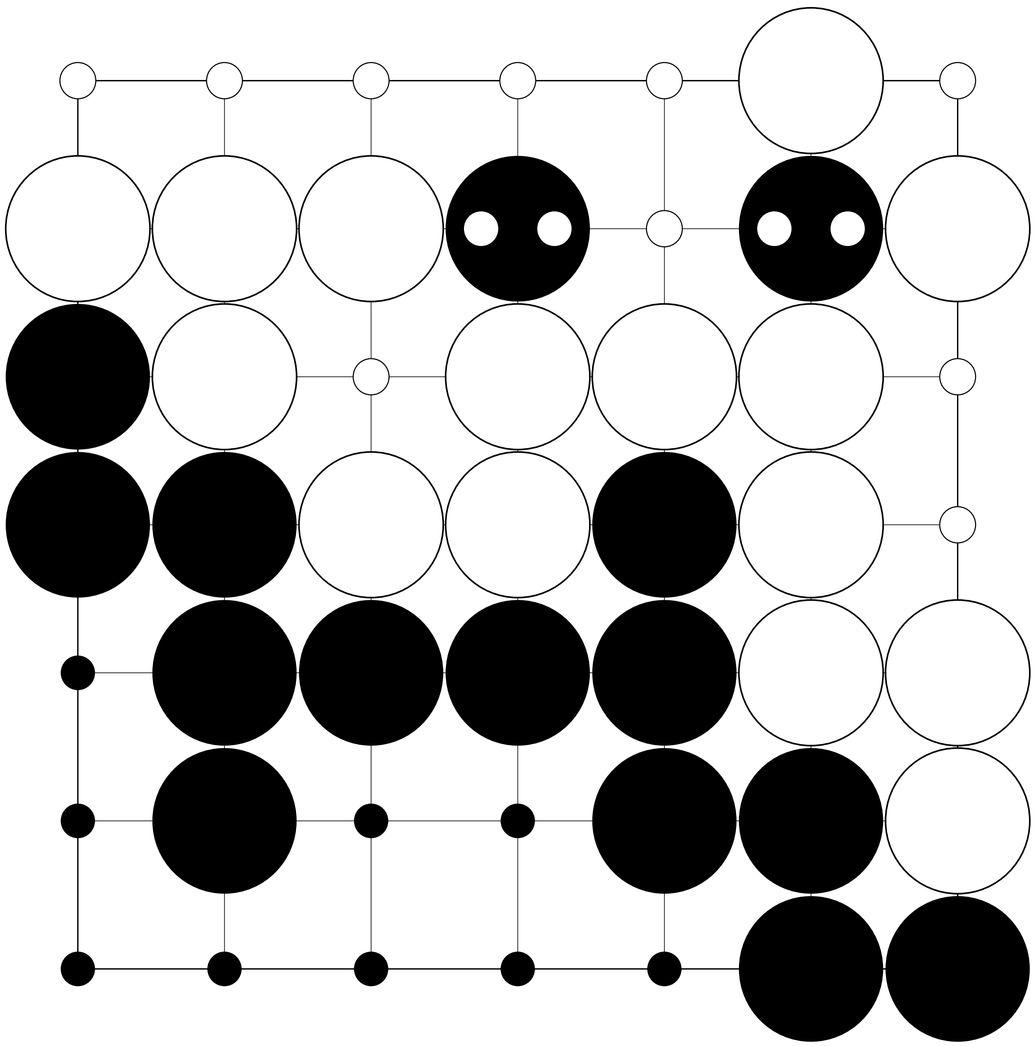
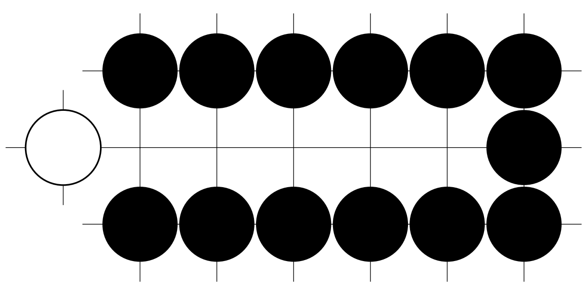




BLOCKED CORRIDORS

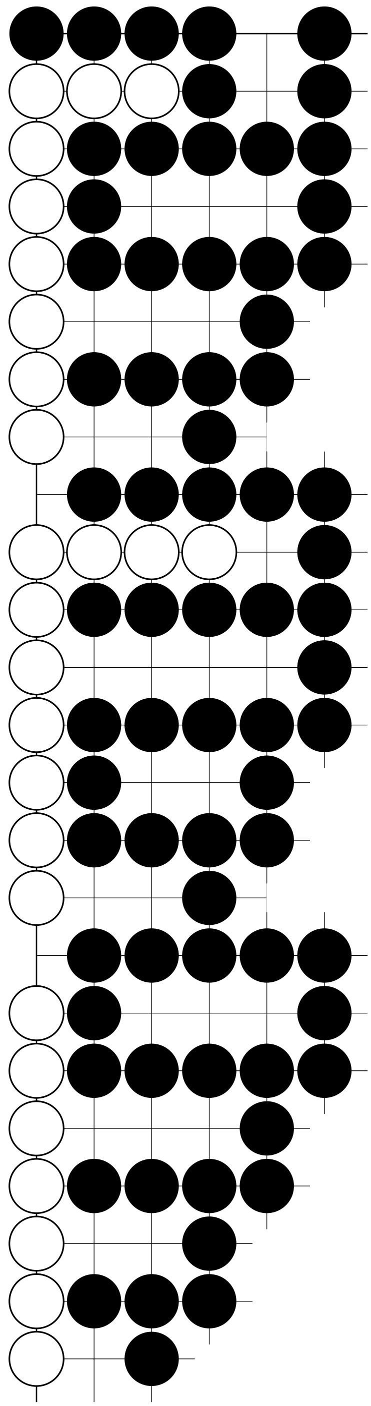
A *blocked corridor* has value

$$n - 2 + \int 2^{1-n}$$



$(n = 5)$

$$3 + \int \frac{1}{16} = \{4 \mid\mid\mid\mid 3 \mid\mid\mid 2 \mid\mid 1 \mid *\}$$



$$\int \left\{ 61\tfrac{3}{4} \;\middle|\middle|\; 37\tfrac{3}{8} \;\middle|\; 18\tfrac{1}{8} \right\}$$

OPEN PROBLEMS

- Are Go endgames EXPTIME-hard (due to ko's)?

- Is Go EXPSPACE-complete under Chinese rules?