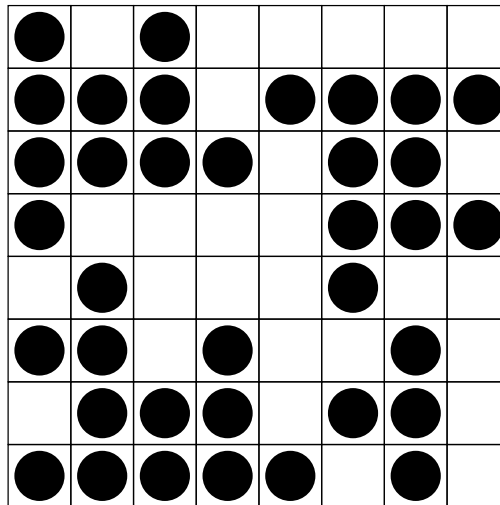


PARITY PARTY WITH PICTURE PROOFS

Erick Knight, David Wolfe

THE PROBLEM

In how many ways can you place checkers on an $n \times n$ checkerboard so that each square is adjacent to an odd number of checkers?

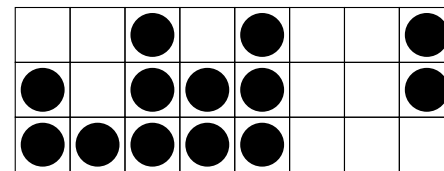
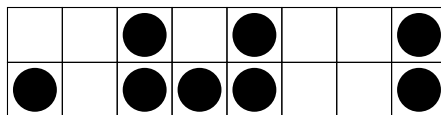


Sources:

P05 from Halici's 2006 puzzleup.com where $n = 8$

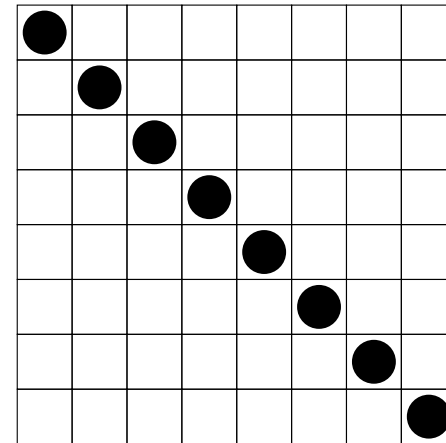
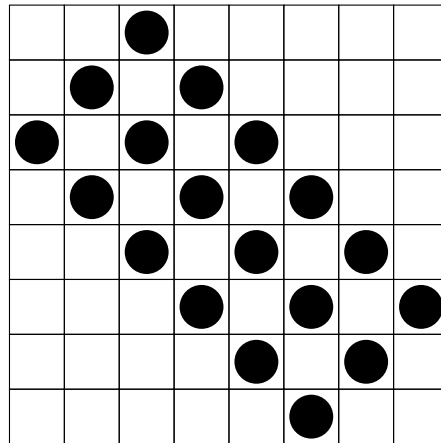
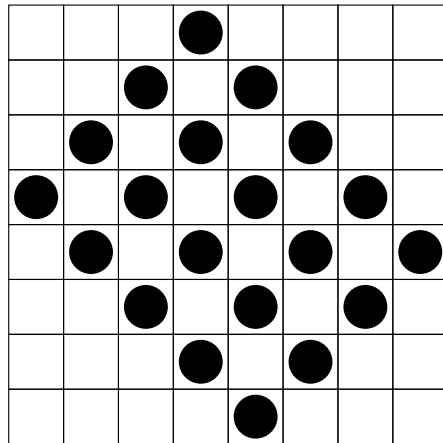
P192 from Vaderlind, Guy, Larson, *The Inquisitive Problem Solver*

TOP ROW DETERMINES LATER ROWS

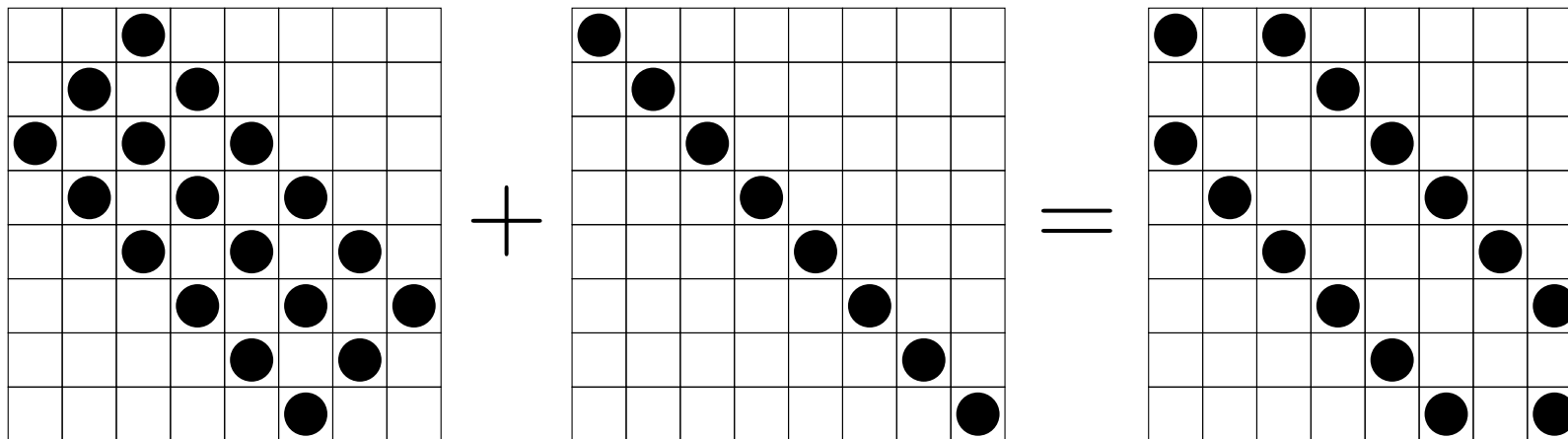


So there can be at most 2^n solutions.

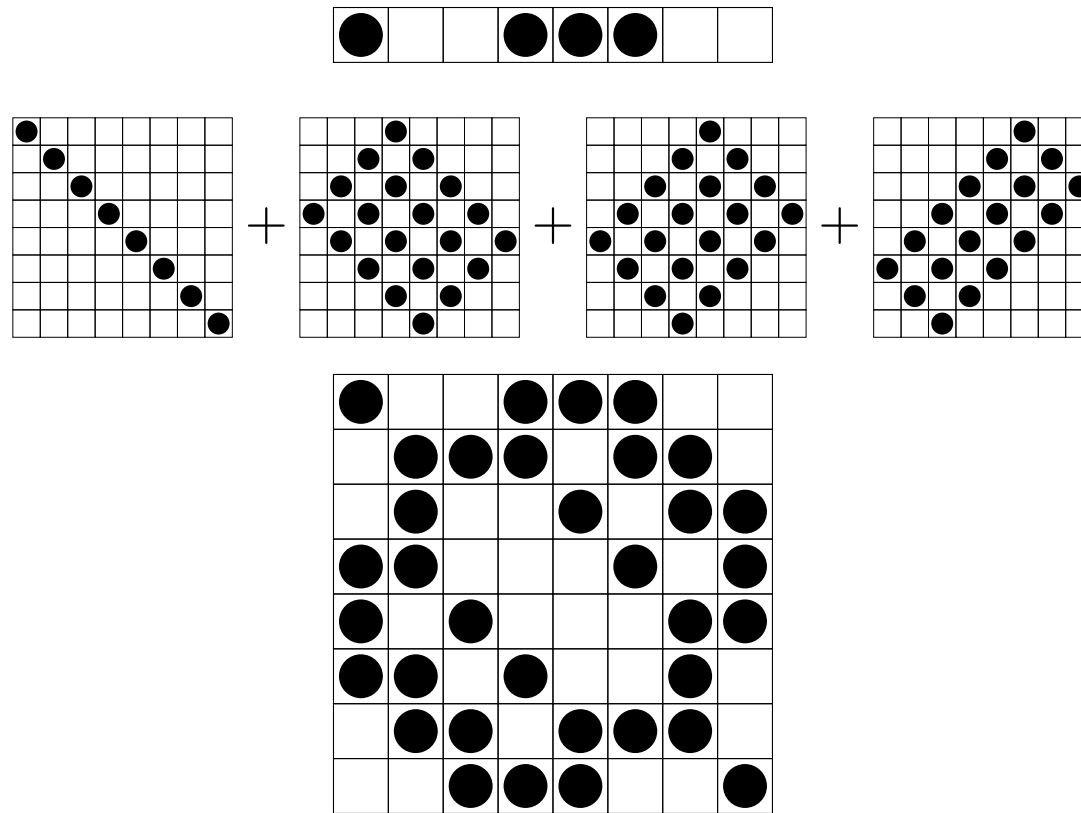
CHANGE ODD TO EVEN



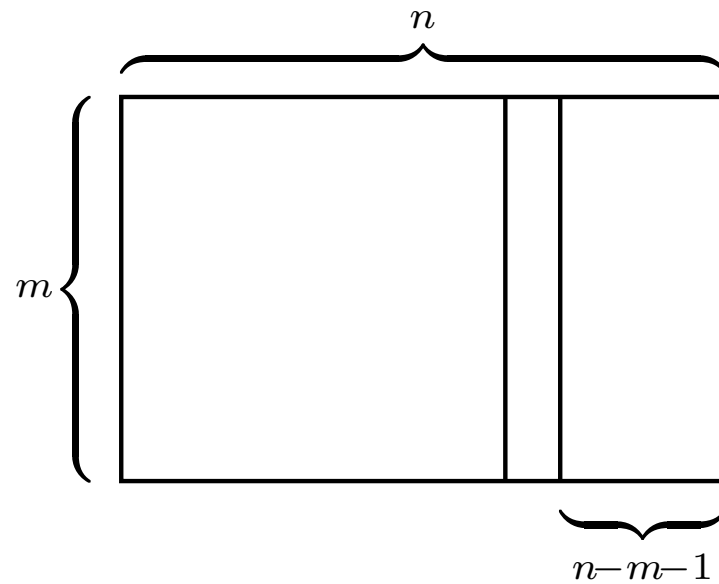
PARITY ADDS!



2^n SOLUTIONS!

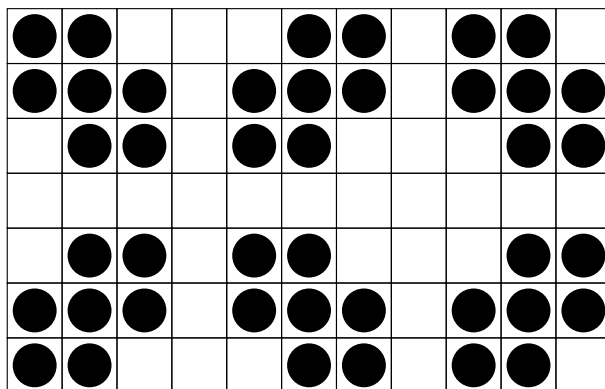


$m \times n$ FOR $m < n$



solutions to $m \times n$ match
solutions to $(n - m - 1) \times n$

$m \times n$

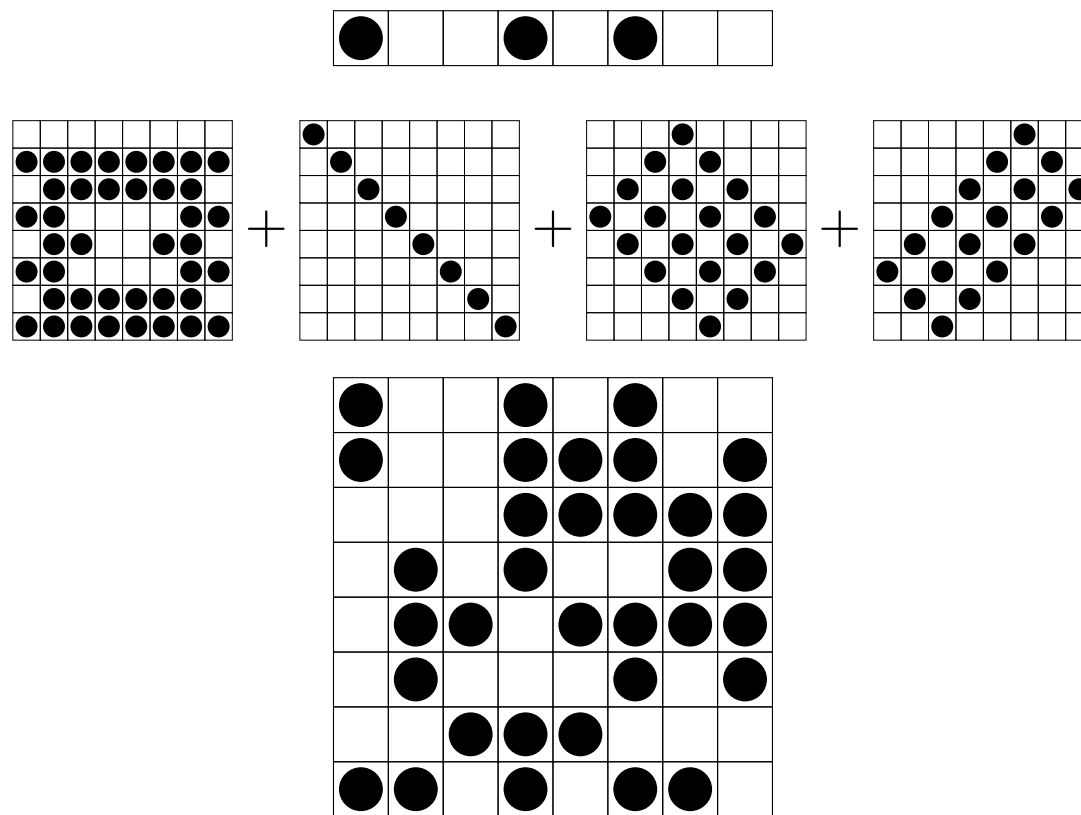


$2^{\gcd(m+1, n+1)-1}$ solutions

BACK TO ODD

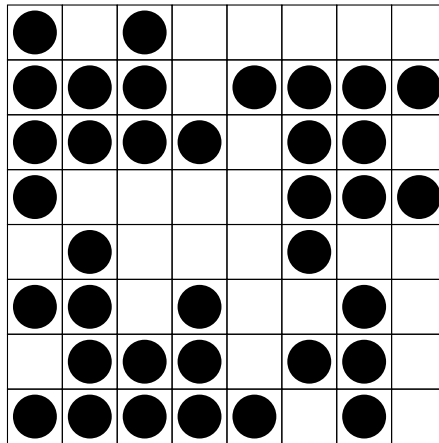
●	●	●	●	●	●	●	●
	●	●	●	●	●	●	
●	●					●	●
	●	●			●	●	
●	●					●	●
	●	●	●	●	●	●	
●	●	●	●	●	●	●	●

ODD PROBLEM: PARITY STILL ADDS!



Either 0 or $2^{\gcd(m+1, n+1)-1}$ solutions

ODD PROBLEM

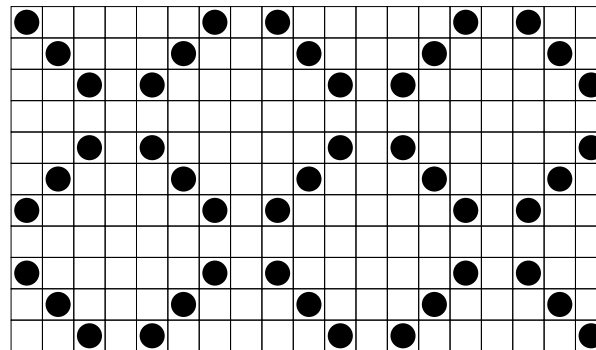


- There must be an even number of checkers.
- Hence, if the even problem has a solution with an odd number of checkers, then the odd problem has no solution.

NO ODD SOLUTIONS

If m and n are odd and end in an equal number of binary 1s, then there are no odd solutions.

$$m = 1011 \quad n = 10011$$



Why? If $d = \gcd(m + 1, n + 1) - 1$, then $d, \frac{m+1}{d+1}, \frac{n+1}{d+1}$ are all odd.

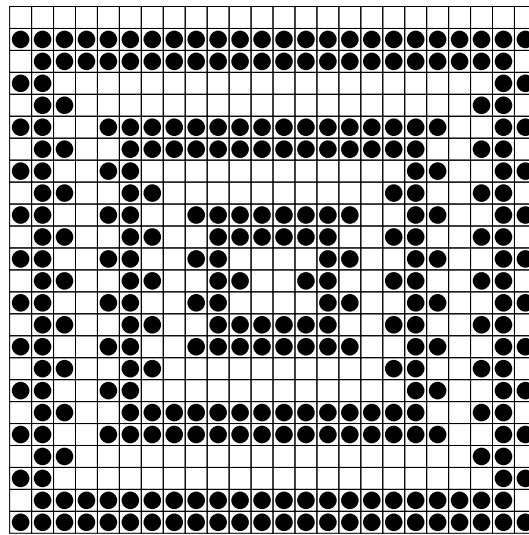
SUMMARY

even problem $\Rightarrow 2^{\gcd(m+1, n+1)-1}$ solutions

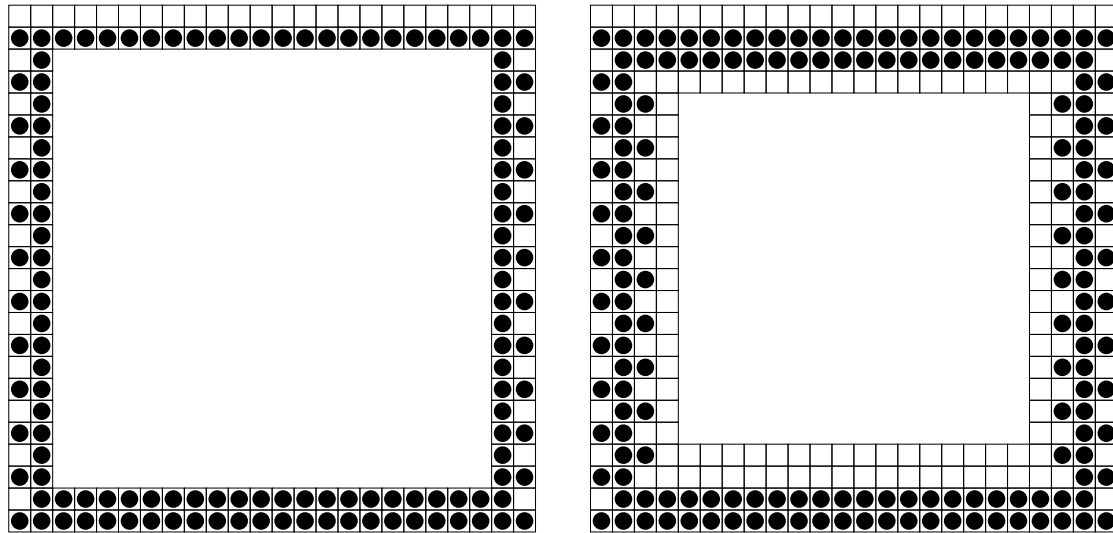
odd problem $\Rightarrow \begin{cases} 0 \text{ solutions} & \text{if } m \text{ and } n \text{ equal oddness} \\ 2^{\gcd(m+1, n+1)-1} & \text{otherwise}^\dagger \end{cases}$

† We now need to prove that there exists a solution!

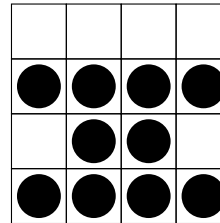
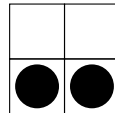
$m \times n$ ODD PROBLEM



$n \times n$ EVEN: INVERT AND INSERT

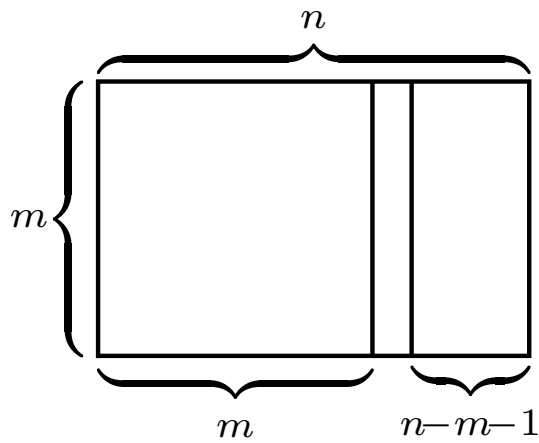


Base cases:

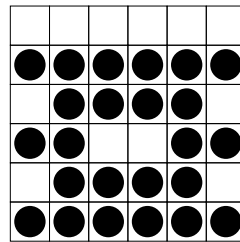


CASE: m OR n EVEN

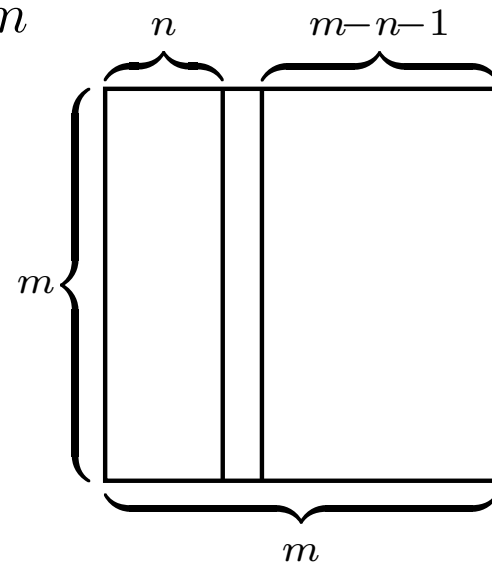
WLOG,
 m even
 n odd or $n \geq m$



$$n > m$$



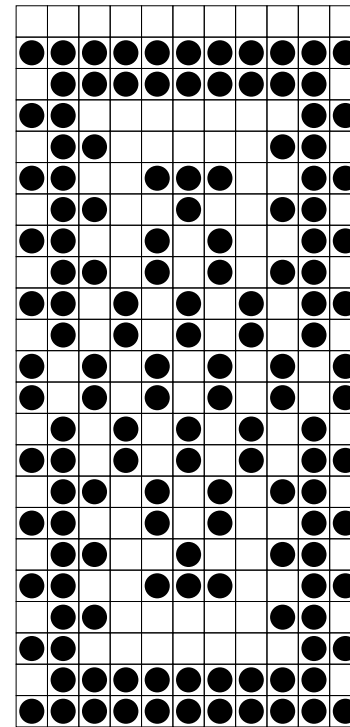
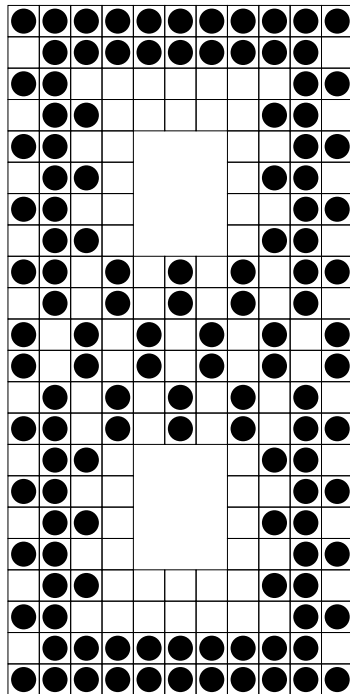
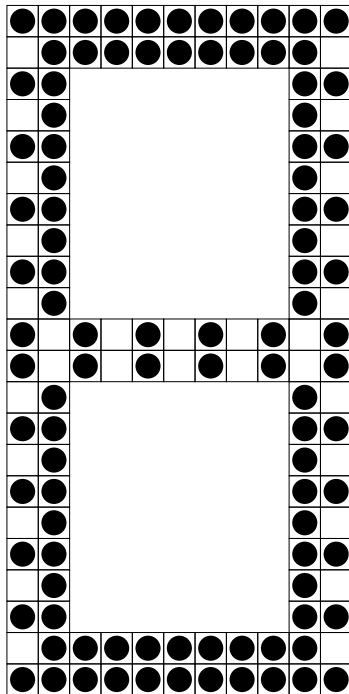
$$n = m \text{ or } n = m - 1$$



$$n < m - 1$$

$$n \text{ odd}$$

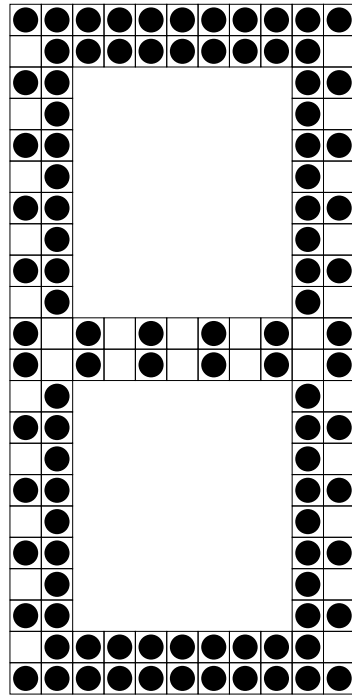
$m \times 2m$, m ODD: INVERT, “HALVE” AND INSERT



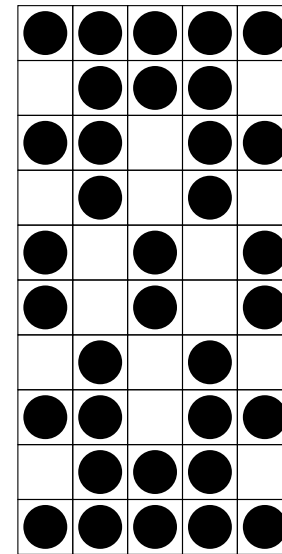
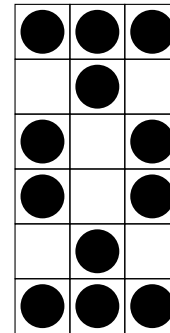
$m \times 2m+1$

$m \times 2m+1$ BOARD HAS 2^m SOLUTIONS

$m \times 2m$ BASE CASES



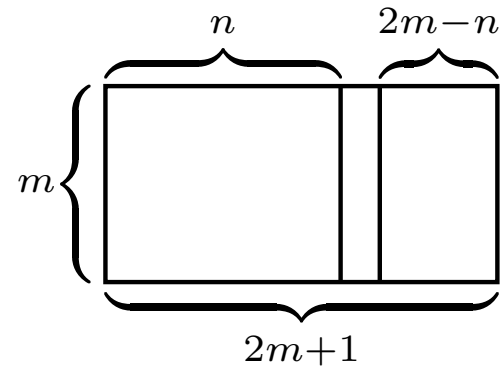
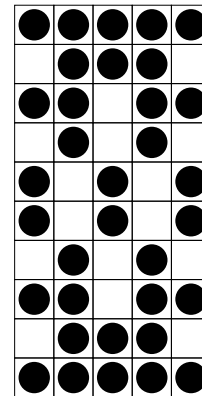
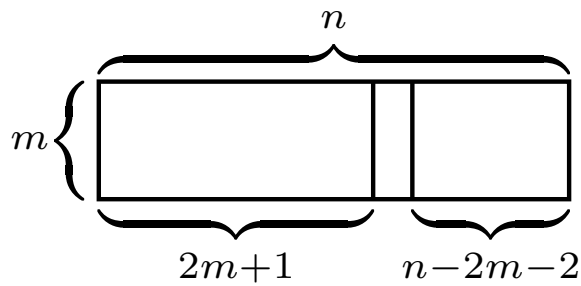
Frame



Base cases

CASE: m AND n ODD

WLOG, $m < n$ with $\mathfrak{o}(m) \neq \mathfrak{o}(n)$



$$n > 2m + 1$$

$$n = 2m + 1$$

$$m < n < 2m + 1$$

To prove: unequal oddness preserved

IF $\mathbf{o}(m) \neq \mathbf{o}(n)$ THEN $\begin{cases} \mathbf{o}(m) \neq \mathbf{o}(n - 2m - 2) \\ \mathbf{o}(m) \neq \mathbf{o}(2m - n) \end{cases}$

m ends 0111

$2m$ ends 01110

$2m + 2$ ends 10000

n	????
$-2m - 2$	<u>-10000</u>
	0111

$2m$	01110
$-n$	<u>-????</u>
	0111

SUMMARY

even problem $\Rightarrow 2^{\gcd(m+1, n+1)-1}$ solutions

odd problem $\Rightarrow \begin{cases} 0 \text{ solutions} & \text{if } m \text{ and } n \text{ equal oddness} \\ 2^{\gcd(m+1, n+1)-1} & \text{otherwise} \end{cases}$