

subset of the space ω^ω of infinite sequences of natural numbers, and consider the following game between Alice and Bob. Alice begins by playing a natural number, then Bob plays another (possibly the same) natural number, then Alice again plays a natural number, and so on. The resulting sequence of moves determines an element $x \in \omega^\omega$. Alice wins if $x \in X$, and Bob wins otherwise. The axiom of determinacy states that this game is determined (i.e., one of the players has a winning strategy) for every choice of X .

A simple construction shows that the axiom of determinacy is inconsistent with the axiom of choice. On the other hand, with Zermelo-Fraenkel set theory plus the axiom of determinacy (ZF+AD), one can prove many non-trivial theorems about the real numbers, including: (i) every subset of \mathbb{R} is Lebesgue measurable; and (ii) every uncountable subset of \mathbb{R} contains a perfect subset. Although ZF+AD is not considered a “realistic” alternative to ZFC (Zermelo-Fraenkel + axiom of choice), it has stimulated a lot of mathematical research, and certain variants of AD are taken rather seriously. For example, the axiom of *projective determinacy* is intimately connected with the continuum hypothesis and the existence of large cardinals (see [10] for details).

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When Multiplication Mixes Up Digits

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My daughter, Lila, has been learning about counting, and I wrote out the digits 1 through 9 on our whiteboard. She asked, “What number is that?”. “Why that’s 123 million, 456 thousand, 789.” “That’s a very big number. Can you make a bigger one?” I doubled the number, and got:

$$123456789 \times 2 = 246913578$$

Wow! The product has the same digits 1 through 9 reordered. Before long, I found myself doubling it over and over again:

$$123456789 \times 2 = 0246913578$$

$$246913578 \times 2 = 0493827156$$

$$493827156 \times 2 = 0987654312$$

$$987654312 \times 2 = 1975308624$$

$$1975308624 \times 2 = 3950617248$$

$$3950617248 \times 2 = 7901234496 \text{ (first exception)}$$

Notice that every product is *pandigital*, until the last product which has two 4's. A pandigital number in base b contains all the base- b digits. The literature varies about whether 0 needs to be one of the digits and whether digits may appear multiple times. Here we say a number is pandigital if it contains either all the digits *or* all the non-zero digits, with no digit repeated.

The fact that multiples of 123456789 and 987654321 are pandigital has long been observed. See, for example, David Wells, *The Penguin Dictionary of Curious and Interesting Numbers*, or the *The Nine Digits* web page.

Why So Many Pandigital Multiples?

It turns out that the doubling process is a red herring, for lots of multiples of 123456789 are pandigital. If you list the numbers under 10 which, when multiplied by 123456789, are pandigital, you find

$$123456789 \times 2 = 246913578$$

$$123456789 \times 4 = 493827156$$

$$123456789 \times 5 = 617283945$$

$$123456789 \times 7 = 864197523$$

$$123456789 \times 8 = 987654312$$

The multipliers are all the single digit numbers which do not have a prime factor of 3.

To investigate what exactly is going on, let's generalize the question to base b :

THEOREM 1. *Let x be the base- b number $123 \dots (b-1)$, and choose an n between 1 and b . The product $n \cdot x$ is pandigital if and only if $b-1$ and n are relatively prime.*

In particular, in base $b = 10$, we have $b-1 = 9$ which has a single prime factor 3. The theorem says that for values of n which *don't* have a factor of 3, i.e., when n is 2, 4, 5, 7, or 8, multiplication by n results in a pandigital product.

Using a diagram, we can compute the product another way by "walking around a clock," and in so doing shed light on the theorem. After describing this clock method, we'll see why it explains the theorem, and then why the clock method correctly computes the product.

Before explaining the clock method, we'll walk through an example of multiplying 123456789 by $n = 4$. On the left below is the usual method for multiplying we learned in grade school:

Grade school method	Drawing the clock	Counting off digits
carries: 01122333 x : 123456789 n : x 4		
product: 493827156	56	27156

The last two digits of the product are $b - n$ (in the example, $b - n = 6$) and $b - n - 1$ (which is 5). To see why, note that the last two digits of x are $b - 1$ and $b - 2$. The last digit of the product is generated by multiplying $n(b - 1) = bn - n = b(n - 1) + (b - n)$, i.e., the last digit will be $b - n$ with a carry of $n - 1$. The second to last digit is generated by the product plus carry $(b - 2)n + (n - 1) = b(n - 1) + (b - n - 1)$, i.e., the second to last digit is $b - n - 1$, with a carry of $n - 1$.

For the clock method, first write the digits 0 through $b - 1$ in a circle. Write down the last two digits of the product $b - n$ and $b - n - 1$ (again, 56 in the example). Cross out the $b - n = 6$. Now, beginning with $b - n - 1 = 5$, count counter-clockwise $n = 4$ positions around the circle to read off the digits in the product. In the example, you'll go from 5 to 1 to 7 to 2 to 8 to 3 to 9 to 4 to 0. Always skip past the crossed out $b - n = 6$ without counting it. Upon recording the 9-digit product, stop; the tenth digit will be 0.

This process hits all the non-zero digits if and only if $b - 1$ (the number of digits *not* crossed out) and n share no common factors. (Otherwise, you'd repeat digits as you used the clock method.)

To see why this alternative way of computing the product works, let us compare what happens when you multiply 123456789 by small numbers like 4 by both the clock method and the grade school method. First, look at the carries. The carry from the last digit is $n - 1 = 3$, and the carries stay the same or decrease proceeding leftward from digit to digit. In our example on the left, $123456789 \cdot 4$, the digit-products with carries are, in order from *right to left*,

$$36, 35, 31, 27, 22, 18, 13, 9, 4$$

Suppose while computing the product by the grade school method, we forgot to carry. Since (working from right to left) the digits of 123456789 decrease by 1, each digit's product by $n = 4$ would decrease by $n = 4$. Reintroducing the carries, since the rightmost digit has no carry but generates a carry of $n - 1 = 3$, the last 2 digits of the product will differ by $n - (n - 1) = 1$. Doing arithmetic mod b , proceeding from right to left, the digits-products decrease by 1 and then by either n or $n + 1$ depending on whether the carry stayed the same or decreased. Further, the carry decreases when the ten's (or, in general, b 's) digit decreases and the units digit increases.

This brings us to why we cross out the 6. Returning to the circle of digits, counting by 4 reflects the fact that consecutive digit products differ by 4 or 5. They differ by 5 when the previous carry decreased, and that's exactly when the previous count around the circle passed 0. If you are currently on digits 7 through 9, the carry must have just dropped, and the next product should decrease by 5 rather than 4. Crossing out the 6 is tantamount to counting down 5 rather than 4 when the current digit is 7 through 9, since the next count will skip the 6.

THEOREM 2. *Let x be the base- b number $(b-1) \dots 321$ and choose an n between 1 and b . The product $n \cdot x$ is pandigital if and only if $b-1$ and n are relatively prime.*

Here, use a slightly different circle process to generate the product. Start with the same circle of digits. Cross out n and write it down as the rightmost digit. Then count by n 's (skipping the crossed out n) *clockwise* until all b digits are recorded.

Returning to Theorem 1, 123456789 times n is pandigital for lots of larger values of n , too. In particular, n can be any of 10, 11, 13, 14, 16, 17, 20, 22, 23, 25, 26, 31, 32, 34, 35, 40, 41, 43 44, 50, 52, 53, 61, 62, 70, 71, or 80. For instance,

$$12345789 \times 71 = 8765432019$$

Note that this list includes no numbers which are a multiple of 3 (which comes as no surprise) but also omits other numbers such as 19. It remains open to generalize this example to base b .

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No Fooling! Newton's Method Can Be Fooled

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You might think that if the Newton sequence of a function converges to a number, that this number must be a zero of the function. At least that's what a group of first semester calculus students thought a couple of years ago.

Most calculus textbooks give examples where the Newton sequence gets stuck (oscillates), hits a horizontal tangent line and fails, or simply converges to a different zero than the one intended, but I don't see calculus textbooks give examples of Newton sequences converging to nonzeros.

Let's look at the Newton iteration and see why the students were so optimistic about the success of Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Typically, the Newton sequence converges to a number L , and the function and its derivative are continuous at L with $f'(L) \neq 0$, so we can let $n \rightarrow \infty$ in the Newton formula, arriving at