Exercises in Asymptotics–Answers

1. Which function/sequence grows faster than the other? [GKP 489]
   (a) \( n^{\ln n} \prec (\ln n)^n \)
   (b) \( n^{\ln\ln n} \prec (\ln n)! \)

2. Let \( f(n) = \sum_{k=1}^{n} \sqrt{k} \).
   (a) Show that \( f(n) = \Theta(n^{3/2}) \), i.e., \( f(n) \asymp n^{3/2} \).
   (b) Find a function/sequence \( g(n) \) for which \( f(n) = g(n) + O(\sqrt{n}) \) and \( f(n) \sim g(n) \). Give support for your answer, of course.

Comparing areas we can deduce
\[
\frac{2}{3} n^{3/2} = \int_0^n \sqrt{x} \, dx < \sum_{k=1}^{n} \sqrt{k} < \int_1^n \sqrt{x} \, dx + \sqrt{n} = \frac{2}{3} n^{3/2} - \frac{2}{3} + \sqrt{n}.
\]

From this we can finish (a) and (b) with \( g(n) = (2/3)n^{3/2} \).

3. By means of Stirling’s approximation, determine what \( \binom{2n}{n} \) is asymptotic to.
   \[
   \binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.
   \]

4. Recall that the \( n^{th} \) harmonic number \( H_n \) is given by
   \[
   H_n = \sum_{k=1}^{n} \frac{1}{k}.
   \]

Justify as much of the following asymptotic approximation as you can:
   \[
   H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).
   \]

Euler’s constant
   \[
   \gamma := \lim_{n \to \infty} (H_n - \ln n)
   \]
   is approximately 0.5772. It is a long-standing open problem to determine whether Euler’s constant is rational or irrational.

For starters, by comparing the areas of inscribed and circumscribed rectangles of base 1 with areas under the graph of \( y = 1/x \), we see \( H_n > \ln(n+1) \) and \( H_n - 1 < \ln n \), so \( \ln(n+1)/n < H_n - \ln n < 1 \ldots \)