Solution of First-Order Linear Recurrence Relations

Given sequences \( \langle a_n \rangle \), \( \langle b_n \rangle \), and \( \langle c_n \rangle \), we shall solve the first-order linear recurrence

\[
a_n Y_n = b_n Y_{n-1} + c_n \quad (n = 1, 2, 3, \ldots)
\]

for \( Y_n \), given the initial value \( Y_0 \). We’ll assume that \( a_n \neq 0 \) and \( b_n \neq 0 \).

Choose the “summation factor” \( s_n \) so that

\[
s_n b_n = s_{n-1} a_{n-1}.
\]

Multiply the given recurrence by \( s_n \), and let \( T_n = s_n a_n Y_n \); we get

\[
T_n = T_{n-1} + s_n c_n.
\]

The solution of this is clearly

\[
T_n = T_0 + \sum_{k=1}^{n} s_k c_k,
\]

where \( T_0 = s_0 a_0 Y_0 = s_1 b_1 Y_0 \), and then \( Y_n = T_n / (s_n a_n) \).

By unwinding/unfolding/backtracking the recurrence

\[
s_n = \frac{s_{n-1} a_{n-1}}{b_n}.
\]

we get a formula for \( s_n \):

\[
s_n = s_1 \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n}.
\]

Therefore

\[
Y_n = \frac{1}{s_n a_n} \left( s_1 b_1 Y_0 + \sum_{k=1}^{n} s_k c_k \right).
\]

In this formula, the \( s_1 \)’s cancel, so we may as well take \( s_1 = 1 \) and for \( n > 1 \) use

\[
s_n = \frac{a_1 a_2 \cdots a_{n-1}}{b_2 b_3 \cdots b_n}.
\]

**Example.** The average number \( C_n \) of comparisons made by quicksort applied to \( n \) items in a random order satisfies \( n C_n = (n + 1) C_{n-1} + 2n \) for \( n > 0 \) and \( C_0 = 0 \). This matches our recurrence with \( a_n = n, b_n = n + 1, c_n = 2n, \) and \( s_n = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{3 \cdot 4 \cdot 5 \cdots (n+1)} = \frac{2}{n(n+1)} \), so

\[
C_n = \frac{1}{n(n+1)} \cdot n \left( 0 + \sum_{k=1}^{n} \frac{2}{k(k+1) \cdot 2k} \right) = 2(n + 1)(H_{n+1} - 1).
\]

References