Moments of a Random Variable

The “moments” of a random variable (or of its distribution) are expected values of powers or related functions of the random variable.

The $r^{th}$ moment of $X$ is $E(X^r)$.

In particular, the first moment is the mean, $\mu_X = E(X)$.

The mean is a measure of the “center” or “location” of a distribution.

Another measure of the “center” or “location” is a median, defined as a value $m$ such that $P(X < m) \leq 1/2$ and $P(X \leq m) \geq 1/2$. If there is only one such value, then it is called the median.

The $r^{th}$ central moment of $X$ is $E[(X - \mu_X)^r]$.

In particular, the second central moment is the variance, $\sigma_X^2 = \text{Var}(X) = E[(X - \mu_X)^2]$. The standard deviation of a random variable is the (nonnegative) square root of the variance: $\sigma_X = \text{Sd}(X) = \sqrt{\sigma_X^2}$.

The variance and standard deviation are measures of the spread or dispersion of a distribution. The standard deviation is measured in the same units as $X$, while the variance is in $X$-units squared.

Another measure of spread is the mean (absolute) deviation: M.A.D. = $E(|X - m|)$ where $m$ is the median of the distribution of $X$. It would seem that this is a more natural measure of spread than the standard deviation or the variance, but it is more difficult to deal with analytically.

The $r^{th}$ factorial moment of $X$ is $E[X^r]$.

Recall that for a positive integer $r$, $X^r = X \cdot (X - 1) \cdot (X - 2) \cdots (X - r + 1) = \prod_{j=1}^{r} (X - j + 1)$.

One measure of skewness is $E[(X - \mu_X)^3]/\sigma_X^3$.

One measure of kurtosis is $E[(X - \mu_X)^4]/\sigma_X^4$.

Properties of the variance. Let $X$ and $Y$ be random variables. Let $a, b, c$ be constants.

- $0 \leq \text{Var}(X) \leq E(X^2)$.
- $\text{Var}(cX) = c^2 \text{Var}(X)$.
- $\text{Var}(X) = E(X^2) - [E(X)]^2$.
- $\text{Var}(aX + bY) = a^2 \text{Var}(X) + 2ab \text{Cov}(X,Y) + b^2 \text{Var}(Y)$.
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$ if $X$ and $Y$ are independent or uncorrelated.