Chapter 9. Planarity

Definitions. A \textit{plane graph} $G$ or a \textit{planar embedding} of $G$ is a drawing of $G$ on the plane in such a way that no two edges meet, except at their common ends. Graphs that do admit such an embedding are called \textit{planar}; ones that don’t are called \textit{nonplanar}. A \textit{girth} of a graph is the length of any smallest cycle in it. An acyclic graph has girth $\infty$. Thus, $3 \leq \text{girth } G \leq \infty$ for any (simple) graph $G$.

Examples of planar graphs are paths, cycles, trees, and the complete bipartite graphs $K_{2,k}$.

Consider a plane graph. A \textit{region} is a maximal connected area that remains when the edges & vertices are removed from the plane. The \textit{boundary} of a region is the vertices and edges touching the region.

\textbf{Theorem} (Jordan Curve Theorem). A simple closed curve partitions the plane into two regions: a bounded interior region and an unbounded exterior region.

\textbf{Lemma} (Lemma A). Any bridge is the boundary of exactly one region. Deleting a bridge (and any resulting isolated vertex) from a plane graph does not change the number of regions. Any nonbridge edge is the boundary of exactly two regions. Deleting a nonbridge edge from a plane graph decreases the number of regions by one.

\textbf{Theorem} (Euler Identity, Theorem 9.1 of CZ). If $G$ is a connected plane graph of order $n$, size $m$, and $r$ regions, then $n - m + r = 2$.

\textit{Proof}. We prove by induction on the number of cycles in $G$. If $G$ has 0 cycle, then $G$ is a tree since $G$ is connected by assumption. Thus, $m = n - 1$ and $r = 1$. Therefore, $n - m + r = n - (n - 1) + 1 = 2$ and the result holds in the base case.

Now let $G$ have $k$ cycles, where $k > 0$, and assume inductively that any connected planar graph having fewer than $k$ cycles satisfies the statement of the Theorem. Let $e$ be an edge belonging to some cycle of $G$. The plane graph $G - e$ has $n$ vertices, $m - 1$ edges, and $r - 1$ regions. Moreover, $G - e$ is connected and has fewer than $k$ cycles. Therefore, by the inductive hypothesis the result holds for $G - e$, i.e., $n - (m - 1) + (r - 1) = 2$. This implies that $n - m + r = 2$, so the result holds for $G$ as well. \hfill $\Box$
**Theorem** (Generalization of Theorem 9.2 of CZ). Let $g$ be a fixed integer $\geq 3$. If $G$ is a planar graph of order $n$, size $m$, girth $\geq g$, and $n \geq (g + 2)/2$, then $m \leq \frac{g(n-2)}{g-2}$.

**Proof.** Note that for any planar graph $G_1$, there is a connected, planar graph $G_2$ that is a supergraph of $G_1$. Thus we may assume that $G$ is connected.

First, assume $G$ has $< g$ edges. Then $G$ is acyclic since it has girth $\geq g$ and so it has too few edges to contain any cycle. Therefore, $G$ is a tree since it’s also connected. Hence, $m = n - 1$. Since $n \geq (g + 2)/2$ by assumption, we have

$$g + 2 \leq 2n$$

i.e.,

$$gn - 2n - g + 2 \leq gn - 2g$$

i.e.,

$$(g - 2)(n - 1) \leq g(n - 2)$$

i.e.,

$$m = n - 1 \leq \frac{g(n - 2)}{g - 2}$$

and the conclusion of the theorem holds.

Next, assume $G$ has $\geq g$ edges. Fix an embedding of $G$ on the plane. For each region $i$ (where $1 \leq i \leq r$) of the plane graph $G$, let $m_i$ be the number of edges on its boundary. Since $G$ has at least $g$ edges, has girth $\geq g$, and is connected, we see that $m_i \geq g$ for each $i$. Thus $\sum_{i=1}^{r} m_i \geq gr$. Also, $\sum_{i=1}^{r} m_i \leq 2m$ because, by Lemma A, each bridge contributes 1 to the sum and each nonbridge contributes 2 to the sum. Thus, $gr \leq 2m$; hence, $r \leq 2m/g$. Combining this last inequality with Euler Identity we have

$$2 = n - m + r \leq n - m + \frac{2m}{g}$$

i.e.,

$$2g \leq gn - (g - 2)m$$

i.e.,

$$(g - 2)m \leq gn - 2g$$

i.e.,

$$m \leq \frac{g(n - 2)}{g - 2}$$

as desired. \qed
Theorem (Theorem 9.2 of CZ). If $G$ is a planar graph of order $n$, size $m$, and $n \geq 3$, then $m \leq 3n - 6$.

Proof. Every graph has girth at least 3. Putting $g = 3$ in the generalized Theorem 9.2 of CZ gives the result. $\square$

Theorem. If $G$ is a bipartite planar graph of order $n$, size $m$, and $n \geq 3$, then $m \leq 2n - 4$.

Proof. A bipartite graph has girth at least 4. Putting $g = 4$ in the generalized Theorem 9.2 of CZ gives the result. $\square$

Theorem (Corollary 9.3 of CZ). Every planar graph contains a vertex of degree $\leq 5$.

Proof. Let $G$ be a planar graph of order $n$ and size $m$. If $n \leq 6$, then every vertex has degree $\leq 5$ and we are done. So assume $n > 6$. By Theorem 9.2, $m \leq 3n - 6$. Thus,

$$\frac{m}{n} \leq 3 - \frac{6}{n}$$

i.e.

$$\frac{2m}{n} \leq 6 - \frac{12}{n}$$

i.e.

$$\frac{2m}{n} < 6$$

since $\frac{12}{n}$ is positive. The last inequality says that the average degree of $G$ is $< 6$. Therefore, there exists at least a vertex whose degree does not exceed the average, i.e., some vertex $v$ has $\deg v \leq \frac{2m}{n} < 6$, i.e., $\deg v \leq 5$. $\square$

Theorem (Corollary 9.4 of CZ). $K_5$ is nonplanar.

Proof. By Theorem 9.2. $\square$

Theorem (Theorem 9.5 of CZ). $K_{3,3}$ is nonplanar.

Proof. By the fact that a bipartite planar graph satisfies $m \leq 2n - 4$. $\square$

Exercise. Show that the Petersen graph is nonplanar by using the generalization of Theorem 9.2.

Definition A subdivision $G'$ of a graph $G$ is a graph that results from inserting one or more vertices of degree 2 into one or more edges of $G$. 

Theorem (Kuratowski’s Theorem). Graph $G$ is planar if and only if $G$ contains no $K_5$ or $K_{3,3}$, or subdivision of $K_5$ or $K_{3,3}$, as a subgraph.

Exercise. Show that the Petersen graph is nonplanar by using Kuratowski’s Theorem.