Homework 1

1. Recall that if \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \) are any two matrices, then their product \( A \cdot B \) is

\[
A \cdot B = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.
\]

Recall also that we define matrix exponentiation by declaring \( A^1 = A \) and for all \( n \geq 1 \), we declare \( A^{n+1} = A \cdot A^n \).

Let \( F \) be the matrix \( \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \). Show by mathematical induction that

\[
F^n = \begin{bmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{bmatrix}
\]

for all \( n = 1, 2, \ldots \), where \( \langle f_0, f_1, f_2, \ldots \rangle \) is the Fibonacci sequence.

2. (a) Find a closed-form formula for

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}
\]

by examining the values of the expression for small values of \( n \).

(b) Prove your formula correct by mathematical induction.

3. Each square of a 1-by-\( n \) chessboard is to be colored red, green, or blue so that no two adjacent squares are colored red. Let \( c_n \) be the number of ways this can be done. Give a recurrence for \( c_n \). Give all necessary base cases and carefully explain how you derive the formula for the recursive case.

4. An arithmetic sequence is a sequence of the form \( \langle a, a+d, a+2d, \ldots, a+(n-1)d, \ldots \rangle \).
   Let \( S(n) \) be the sum of the first \( n \) terms of an arithmetic sequence.
(a) Give a closed-form formula for $S(n)$ (in terms of $a$, $d$, and $n$).

(b) $-10 - 6 - 2 + 2 + 6 + \cdots + 102 = ?$

5. A geometric sequence is a sequence of the form $\langle a, ar, ar^2, \ldots, ar^{n-1}, \ldots \rangle$. Let $T(n)$ be the sum of the first $n$ terms of a geometric sequence.

(a) Give a closed-form formula for $T(n)$ if $r = 1$.

(b) Let $r \neq 1$. Show by mathematical induction that

$$T(n) = a \frac{1 - r^n}{1 - r} \quad \text{or, equivalently} \quad a \frac{r^n - 1}{r - 1}.$$

(c) $3/5 + 3/25 + 3/125 + \cdots + 3/25^{10} = ?$

6. Suppose you begin with a pile of $n$ stones and split this pile into $n$ piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have $r$ and $s$ stones in them, respectively, you compute $rs$. Show that no matter how you split the piles, the sum of the product computed at each step equals $n(n - 1)/2$. (Hint: Use strong induction.)