Parity Party with Picture Proofs

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In how many ways can you place checkers on an $n \times n$ checkerboard so that each square is adjacent to an odd number of checkers?

Sources:
P05 from Halici’s 2006 puzzleup.com where $n = 8$
P192 from Vaderlind, Guy, Larson, *The Inquisitive Problem Solver*
Top row determines later rows

So there can be at most $2^n$ solutions.
Change odd to even
Parity adds!
$2^n$ SOLUTIONS!
\( m \times n \) for \( m < n \)

solutions to \( m \times n \) match
solutions to \( (n - m - 1) \times n \)
$m \times n$

$2^{\gcd(m+1,n+1)} - 1$ solutions
Back to odd
Odd problem: parity still adds!

Either 0 or $2^{\gcd(m+1,n+1)-1}$ solutions
Odd problem

- There must be an even number of checkers.
- Hence, if the even problem has a solution with an odd number of checkers, then the odd problem has no solution.
If \( m \) and \( n \) are odd and end in an equal number of binary 1s, then there are no odd solutions.

\[
m = 1011 \quad n = 10011
\]

Why? If \( d = \gcd(m + 1, n + 1) - 1 \), then \( d, \frac{m+1}{d+1}, \frac{n+1}{d+1} \) are all odd.
Summary

**even problem** ⇒ $2^{\text{gcd}(m+1,n+1)-1}$ solutions

**odd problem** ⇒ $\begin{cases} 0 \text{ solutions} & \text{if } m \text{ and } n \text{ equal oddness} \\ 2^{\text{gcd}(m+1,n+1)-1} & \text{otherwise} \end{cases}$†

†We now need to prove that there exists a solution!
$m \times n$ ODD PROBLEM
$n \times n$ EVEN: INVERT AND INSERT

Base cases:
**Case: m or n even**

WLOG,

\( m \) even

\( n \) odd or \( n \geq m \)

\( n > m \)

\( n = m \) or \( n = m - 1 \)

\( n < m - 1 \)

\( n \) odd
$m \times 2m$, $m$ ODD: INVERT, “HALVE” AND INSERT

$m \times 2m + 1$ BOARD HAS $2^m$ SOLUTIONS
$m \times 2m$ BASE CASES

Frame

Base cases
Case: $m$ and $n$ odd

WLOG, $m < n$ with $o(m) \neq o(n)$

To prove: unequal oddness preserved
IF \( o(m) \neq o(n) \) THEN

\[
\begin{cases}
  o(m) \neq o(n - 2m - 2) \\
  o(m) \neq o(2m - n)
\end{cases}
\]

\[m\] ends 0111

\[2m\] ends 01110

\[2m + 2\] ends 10000

\[
\begin{array}{c|c}
  n & \text{???} \\
  \hline
  -2m - 2 & -10000 \\
  \hline
  0111
\end{array}
\]

\[
\begin{array}{c|c}
  2m & 01110 \\
  \hline
  -n & -???? \\
  \hline
  0111
\end{array}
\]
Summary

**even** problem $\Rightarrow 2^{\gcd(m+1,n+1)-1}$ solutions

**odd** problem $\Rightarrow \begin{cases} 
0 \text{ solutions} & \text{if } m \text{ and } n \text{ equal oddness} \\
2^{\gcd(m+1,n+1)-1} & \text{otherwise}
\end{cases}$