DISCRETE GRONWALL LEMMA AND APPLICATIONS

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Gronwall's lemma states an inequality that is useful in the theory of differential equations. Here is one version of it [1, p, 283]:

0. Gronwall's lemma. Let $y(t)$, $f(t)$, and $g(t)$ be nonnegative functions on $[0, T]$ having one-sided limits at every $t \in [0, T]$, and assume that for $0 \le t \le T$ we have

$$y(t) \le f(t) + \int_0^t g(s)y(s)\,ds.$$

Then for $0 \le t \le T$ we also have

$$y(t) \le f(t) + \int_0^t g(s)f(s)\exp\left(\int_s^t g(u)\,du\right)\,ds.$$

Proof. Let $I(t) := \int_0^t g(s)y(s)\,ds$ and $G(t) := \int_0^t g(s)\,ds$. Now $I'(t) = g(t)y(t) \le g(t)f(t) + g(t)I(t)$ outside a denumerable set, and $\frac{d}{dt}e^{-G(t)}I(t) = e^{-G(t)}\{I'(t) - g(t)I(t)\} \le e^{-G(t)}g(t)f(t)$, so $e^{-G(t)}I(t) \le \int_0^t e^{-G(s)}g(s)f(s)\,ds$. Therefore, $I(t) \le \int_0^t e^{G(t)-G(s)}g(s)f(s)\,ds$ and $y(t) \le f(t) + I(t) \le f(t) + \int_0^t e^{G(t)-G(s)}g(s)f(s)\,ds$. $\square$

Here is our first formulation of a discrete version of this lemma:

1. Discrete Gronwall inequality. If $\langle y_n\rangle$, $\langle f_n\rangle$, and $\langle g_n\rangle$ are nonnegative sequences and

$$y_n \le f_n + \sum_{0\le k<n} g_k y_k \quad \text{for} \quad n \ge 0, \tag{1}$$

then

$$y_n \le f_n + \sum_{0\le k<n} f_k g_k \exp\Big(\sum_{k<j<n} g_j\Big) \quad \text{for} \quad n \ge 0. \tag{2}$$

The proof that follows first gives the exact solution for y_n when inequality in (1) is replaced by equality. Then it shows that any solution of the inequality is dominated by this exact solution, from which (2) quickly follows. First we prove a lemma giving a solution of the equality when $f_n \equiv 1$.

2. Lemma. Let $\langle g_n\rangle$ be a sequence. For $n \ge 0$ let

$$G_n := \prod_{0\le j<n}(1 + g_j). \tag{3}$$

Then

$$G_n = 1 + \sum_{0 \le j < n} g_j G_j$$

for $n \ge 0$, and, more generally, for $0 \le k \le n$,

$$G_n = G_k + \sum_{k \le j < n} g_j G_j. \tag{4}$$

Proof. An empty product is 1, and an empty sum is 0, so the formulas hold for $n = 0$ (and $k = 0$). Let $m \ge 0$. Assuming the formulas hold for $n = m$, we have $G_{m+1} = (1 + g_m)G_m = G_m + g_m G_m = 1 + \sum_{0 \le k < m} g_k G_k + g_m G_m = 1 + \sum_{0 \le k < m+1} g_k G_k$, and for $0 \le k \le m$, $G_{m+1} = G_m + g_m G_m = G_k + \sum_{k \le j < m} g_j G_j + g_m G_m = G_k + \sum_{k \le j < m+1} g_j G_j$. The conclusions follow by mathematical induction. □

3. Proposition. Assume $\langle x_n \rangle$, $\langle f_n \rangle$, and $\langle g_n \rangle$ are nonnegative sequences and

$$x_n = f_n + \sum_{0 \le k < n} g_k x_k \quad \text{for} \quad n \ge 0. \tag{5}$$

Then

$$x_n = f_n + \sum_{0 \le k < n} f_k g_k \prod_{k < j < n} (1 + g_j) = f_n + \sum_{0 \le k < n} f_k g_k G_n / G_{k+1} \tag{6}$$

for $n \ge 0$. Here G_n is defined by (3).

Proof. Let $\chi_n := f_n + \sum_{0 \le k < n} f_k g_k G_n / G_{k+1}$. We prove that $x_n = \chi_n$ for $n \ge 0$ by induction. For $n = 0$ both sides equal f_0. Let $m > 0$. Assume $x_n = \chi_n$ for $0 \le n < m$. By

(5),

$$
\begin{aligned}
x_m &= f_m + \sum_{0\le k<m} g_k x_k = f_m + \sum_{0\le k<m} g_k \chi_k \\
&= f_m + \sum_{0\le k<m} g_k\{f_k + \sum_{0\le j<k} f_j g_j G_k/G_{j+1}\} \\
&= f_m + \sum_{0\le k<m} g_k f_k + \sum_{0\le k<m}\sum_{0\le j<k} g_k f_j g_j G_k/G_{j+1} \\
&= f_m + \sum_{0\le k<m} g_k f_k + \sum_{0\le j<m-1}\sum_{j<k<m} g_k f_j g_j G_k/G_{j+1} \\
&= f_m + \sum_{0\le k<m} g_k f_k + \sum_{0\le k<m-1}\sum_{k<j<m} g_j f_k g_k G_j/G_{k+1} \\
&= f_m + f_{m-1}g_{m-1} + \sum_{0\le k<m-1} \{f_k g_k(1 + \sum_{k<j<m} g_j G_j/G_{k+1})\} \\
&= f_m + \sum_{0\le k<m} f_k g_k G_m/G_{k+1} = \chi_m
\end{aligned}
$$

by (4). The conclusion follows by induction. □

The following theorem actually gives a sharper result than the discrete Gronwall inequality.
4. Theorem: Sharp Gronwall Inequality. Assume $\langle y_n\rangle$, $\langle f_n\rangle$, and $\langle g_n\rangle$ are nonnegative sequences and

(7)
$$y_n \le f_n + \sum_{0\le k<n} g_k y_k \quad \text{for} \quad n \ge 0.$$

Then

(8)
$$y_n \le f_n + \sum_{0\le k<n} f_k g_k \prod_{k<j<n} (1+g_j)$$

for $n \ge 0$.

Proof. Let x_n be as in the previous proposition. Then an easy induction shows that $y_n \le x_n$ for every $n \ge 0$. □

Alternative Proof. For $n = 0$ inequality (7) implies $y_0 \le f_0$, whence inequality (8) holds for $n = 0$. Let $m > 0$. Assume that (8) holds for $0 \le n < m$. By hypothesis (7) and this induction hypothesis,

$$\begin{aligned}
y_m &\le f_m + \sum_{0\le k<m} g_k y_k \\
&\le f_m + \sum_{0\le k<m} g_k \left(f_k + \sum_{0\le j<k} f_j g_j \prod_{j<i<k} (1+g_i) \right) \\
&= f_m + \sum_{0\le j<m} f_j g_j G_j^{(m)}
\end{aligned}$$

where (by an implicit induction)

$$\begin{aligned}
G_j^{(m)} &= 1 + \sum_{j<k<m} g_k \prod_{j<i<k} (1+g_i) \\
&= 1 + g_{j+1}\cdot 1 + g_{j+2}(1+g_{j+1}) + \cdots + g_{m-1}(1+g_{j+1})\cdots(1+g_{m-2}) \\
&= (1+g_{j+1})(1+g_{j+2}+\cdots+g_{m-1}(1+g_{j+2})\cdots(1+g_{m-2})) \\
&= \cdots \\
&= \prod_{j<i<m} (1+g_i).
\end{aligned}$$

Thus inequality (8) holds for $n = m$.
By mathematical induction, inequality (8) holds for every $n \ge 0$. $\square$

Proof of the Discrete Gronwall inequality. Use the inequality $1 + g_j \le \exp(g_j)$ in the previous theorem. $\square$

5. Another discrete Gronwall inequality

Here is another form of Gronwall's lemma that is sometimes invoked in differential equations [2, pp. 48–49]:
Let y and g be nonnegative integrable functions and c a nonnegative constant. If

$$y(t) \le c + \int_0^t g(s)y(s)\,ds \qquad \text{for } t \ge 0,$$

then

$$y(t) \le c\exp\left(\int_0^t g(s)\,ds\right) \qquad \text{for } t \ge 0.$$

Proposition: Special Gronwall Inequality. Let $\langle y_n\rangle$ and $\langle g_n\rangle$ be nonnegative sequences and c a nonnegative constant. If

$$y_n \le c + \sum_{0\le k<n} g_k y_k \qquad \text{for } n \ge 0,$$

then

$$y_n \le c \prod_{0\le j<n} (1+g_j) \le c\exp\left(\sum_{0\le j<n} g_j\right) \qquad \text{for } n \ge 0.$$

Proof. Assume the hypothesis and apply the Sharp Gronwall Inequality with $f_n \equiv c$:

$$\begin{aligned}
y_n &\le c + \sum_{0\le k<n} c g_k \prod_{k<j<n} (1+g_j) \\
&= c + c \sum_{0\le k<n} \{ \prod_{k\le j<n} (1+g_j) - \prod_{k+1\le j<n} (1+g_j)\} \\
&= c + c\{ \prod_{0\le j<n} (1+g_j) - \prod_{n\le j<n} (1+g_j)\} \qquad \text{[telescoping sum]} \\
&= c \prod_{0\le j<n} (1+g_j) \qquad \text{[empty product is 1]} \\
&\le c \exp\left(\sum_{0\le j<n} g_j \right) \qquad [1+g_j \le e^{g_j}].\ \square
\end{aligned}$$

Nonhomogeneous Linear Systems

One area where Gronwall's inequality is used is the study of the asymptotic behavior of nonhomogeneous linear systems of differential equations. We are interested in obtaining discrete analogs.

6. First-order nonhomogeneous linear differential equations

Consider the first-order nonhomogenous vector differential equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$$

for $t \ge 0$ with initial condition $\mathbf{x}(0) = \mathbf{c}$, where $A(t)$ is a continuous $d \times d$ matrix, $\mathbf{f}(t)$ is a continuous $d \times 1$ vector, and $\mathbf{x}(t)$ and $\mathbf{c}$ are a $d \times 1$ vectors. Its solution may be obtained in terms of d linearly independent solutions of the corresponding homogeneous equation as follows:

$$\mathbf{x}(t) = Y(t)\mathbf{c} + Y(t) \int_0^t Y(s)^{-1} \mathbf{f}(s)\, ds$$

where $Y(t)$ is the $d \times d$ matrix solution of $\dfrac{dY}{dt} = A(t)Y$ with $Y(0) = I$. [2, pp. 91-92]

7. First-order nonhomogeneous linear difference equations

We prove the discrete counterpart. For vector sequences $\langle \mathbf{x}(n) \rangle$ the forward difference operator Δ is defined by: $\Delta\mathbf{x}(n) = \mathbf{x}(n+1) - \mathbf{x}(n)$. The analog of the differential equation is

$$\Delta\mathbf{x}(n) = A(n)\mathbf{x}(n) + \mathbf{f}(n),$$

or

$$\mathbf{x}(n+1) = (I + A(n))\mathbf{x}(n) + \mathbf{f}(n)$$

for $n \ge 0$ with $\mathbf{x}(0) = \mathbf{c}$.

Proposition. Let $\langle \mathbf{x}(n) \rangle$ and $\langle \mathbf{f}(n) \rangle$ be sequences of $d \times 1$ vectors, let $\mathbf{c}$ be a fixed $d \times 1$ vector, and let $\langle A(n) \rangle$ be a sequence of $d \times d$ matrices. Assume that $I + A(n)$ is invertible for each $n \geq 0$. If

$$\Delta \mathbf{x}(n) = A(n)\mathbf{x}(n) + \mathbf{f}(n) \qquad \text{for } n \geq 0$$

and $\mathbf{x}(0) = \mathbf{c}$, and if $Y(n)$ is the $d \times d$ solution of $\Delta Y(n) = A(n)Y(n)$ with $Y(0) = I$, then

$$\mathbf{x}(n) = Y(n)\mathbf{c} + Y(n) \sum_{0 \leq k < n} Y(k+1)^{-1}\mathbf{f}(k) \qquad \text{for } n \geq 0. \tag{9}$$

Proof. Assuming $I + A(n)$ is invertible for $n \geq 0$, we let $Z(n)$ be the matrix solution of the "adjoint" equation

$$\Delta Z(n) = -Z(n+1)A(n) \qquad \text{for } n \geq 0 \text{ with } Z(0) = I.$$

Then $(\Delta Z(n))Y(n) = -Z(n+1)A(n)Y(n)$ and $Z(n+1)\Delta Y(n) = Z(n+1)A(n)Y(n)$, so $(\Delta Z(n))Y(n) + Z(n+1)A(n)Y(n) = 0$, the zero matrix, i.e., $\Delta(Z(n)Y(n)) = 0$. Therefore $Z(n)Y(n) = C$, a constant matrix. Because $Z(0) = I$ and $Y(0) = I$, we have $Z(n)Y(n) = I$, whence $Z(n) = Y(n)^{-1}$ for $n \geq 0$. Now multiply $\Delta \mathbf{x}(n) = A(n)\mathbf{x}(n) + \mathbf{f}(n)$ on the left by $Z(n+1)$ to get

$$Z(n+1)\Delta \mathbf{x}(n) = Z(n+1)A(n)\mathbf{x}(n) + Z(n+1)\mathbf{f}(n),$$

multiply $\Delta Z(n) = -Z(n+1)A(n)$ on the right by $\mathbf{x}(n)$ to get

$$(\Delta Z(n))\mathbf{x}(n) = -Z(n+1)A(n)\mathbf{x}(n),$$

and add to get

$$Z(n+1)\Delta \mathbf{x}(n) + (\Delta Z(n))\mathbf{x}(n) = Z(n+1)\mathbf{f}(n)$$

for $n \geq 0$, i.e.,

$$\Delta(Z(n)\mathbf{x}(n)) = Z(n+1)\mathbf{f}(n).$$

Therefore,

$$\sum_{0 \leq k < n} \Delta(Z(k)\mathbf{x}(k)) = \sum_{0 \leq k < n} Z(k+1)\mathbf{f}(k),$$

so

$$Z(n)\mathbf{x}(n) - Z(0)\mathbf{x}(0) = \sum_{0 \leq k < n} Z(k+1)\mathbf{f}(k).$$

Since $Z(0) = I$ and each $Z(n) = Y(n)^{-1}$,

$$\mathbf{x}(n) = Y(n)\mathbf{x}(0) + Y(n) \sum_{0 \leq k < n} Y(k+1)^{-1}\mathbf{f}(k). \quad \square$$

8. Matrix differential equations and linearly independent solutions

In the case of the matrix differential equation $\dfrac{dX}{dt} = A(t)X(t)$ one can show that $\dfrac{d|X|}{dt} = (\text{trace } A(t))|X(t)|$, where $|X(t)| = \det X(t)$, whence $|X(t)| = |X(0)| \exp\left(\int_0^t \text{trace } A(s)\, ds\right)$. [2, p. 88] From this it follows that a set of d linearly independent vector solutions of $\dfrac{d\mathbf{x}}{dt} = A(t)\mathbf{x}$ will remain linearly independent as time goes on.

9. Matrix difference equations and linearly independent solutions

The discrete analog does not play out in the same neat way, but it is even simpler. If the sequence $\langle X(n) \rangle$ of $d \times d$ matrices satisfies $\Delta X(n) = A(n)X(n)$ for $n \geq 0$, then $X(n+1) = (I + A(n))X(n)$ for $n \geq 0$, so that $X(n) = (I + A(n-1))(I + A(n-2)) \cdots (I + A(0))X(0)$ for

$n > 0$. Thus, if each $I + A(k)$ is invertible, then $X(n)$ is nonsingular whenever $X(0)$ is, and so a set of d linearly independent vector solutions of $\Delta\mathbf{x}(n) = A(n)\mathbf{x}(n)$ will remain linearly independent as time n goes on.

Let $\| \cdot \|$ denote a norm for d-vectors and the corresponding operator norm for $d \times d$ matrices. For example, $\| \mathbf{x} \|$ and $\| A \|$ could be defined as the sums of the absolute values of their components.

10. A differential equation application of Gronwall's inequality

One application of Gronwall's inequality is in the proof of the following theorem. [2, pp. 119-120]

Theorem. Assume $\lim_{t\to\infty} A(t) = A$, a constant matrix. If all the solutions of the limiting homogeneous equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

remain bounded as $t \to \infty$, then the same is true of all solutions of the nonhomogeneous equation

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$$

provided that

$$\int_0^\infty \| A(t) - A \| \; dt < \infty$$

and

$$\int_0^\infty \| \mathbf{f}(t) \| \; td < \infty.$$

11. A difference equation application of Gronwall's inequality

Our analogous theorem is the following.

Theorem. Assume that $\lim_{n\to\infty} A(n) = A$, a constant matrix. If all the solutions of the limiting homogeneous equation

$$\Delta\mathbf{x}(n) = A\mathbf{x}(n) \qquad \text{for } n \geq 0$$

remain bounded as $n \to \infty$, then the same is true of all solutions of the nonhomogeneous equation

$$\Delta\mathbf{x}(n) = A(n)\mathbf{x}(n) + \mathbf{f}(n) \qquad \text{for } n \geq 0 \tag{10}$$

provided that

$$\sum_{n=0}^{\infty} \| A(n) - A \| < \infty$$

and

$$\sum_{n=0}^{\infty} \| \mathbf{f}(n) \| < \infty.$$

Proof. Let $B(n) = A(n) - A$ and rewrite (10) as

$$\Delta\mathbf{x}(n) = A\mathbf{x}(n) + B(n)\mathbf{x}(n) + \mathbf{f}(n).$$

By (9) with $\mathbf{f}(n)$ replaced by $B(n)\mathbf{x}(n)+\mathbf{f}(n)$, we find

$$\mathbf{x}(n) = Y(n)\mathbf{x}(0) + Y(n)\sum_{0\le k<n} Y(k+1)^{-1}(B(k)\mathbf{x}(k)+\mathbf{f}(k))$$

where $\Delta Y(n) = AY(n)$ for $n \ge 0$ and $Y(0) = I$. Therefore $Y(n) = (I+A)^n$ for $n \ge 0$ and $Y(n)Y(k+1)^{-1} = (I+A)^n(I+A)^{-(k+1)} = (I+A)^{n-k-1} = Y(n-k-1)$ and

$$\mathbf{x}(n) = Y(n)\mathbf{x}(0) + \sum_{0\le k<n} Y(n-k-1)(B(k)\mathbf{x}(k)+\mathbf{f}(k)).$$

By hypothesis, every $\| Y(n) \| \le c_1$, a constant, so, upon taking norms, we get

$$\begin{aligned}
\| \mathbf{x}(n) \| &\le \| Y(n) \| \cdot \| \mathbf{x}(0) \| + \sum_{0\le k<n} \| Y(n-k-1) \| (\| B(k) \| \cdot \| \mathbf{x}(k) \| + \| \mathbf{f}(k) \|) \\
&\le c_1 \| \mathbf{x}(0) \| + c_1 \sum_{0\le k<n} \| \mathbf{f}(k) \| + \sum_{0\le k<n} c_1 \| B(k) \| \cdot \| \mathbf{x}(k) \| \\
&= f_n + \sum_{0\le k<n} g_k \| \mathbf{x}(k) \|
\end{aligned}$$

where f_n is the sum of the first two terms and $g_k = c_1 \| B(k) \|$. Here

$$\begin{aligned}
f_n &= c_1 \| \mathbf{x}(0) \| + c_1 \sum_{0\le k<n} \| \mathbf{f}(k) \| \\
&\le c_1 \| \mathbf{x}(0) \| + c_1 \sum_{0\le k<\infty} \| \mathbf{f}(k) \| =: c_2 < \infty,
\end{aligned}$$

so

$$\| \mathbf{x}(n) \| \le c_2 + \sum_{0\le k<n} c_1 \| B(k) \| \cdot \| \mathbf{x}(k) \| .$$

By the Special Gronwall Inequality,

$$\| \mathbf{x}(n) \| \le c_2 \exp\left(\sum_{0\le k<n} c_1 \| B(k) \|\right) \le c_2 \exp\left(\sum_{0\le k<\infty} c_1 \| A(k) - A \|\right) < \infty,$$

showing that the solutions are bounded. $\square$

References

[1] Dieudonné, J., *Foundations of Modern Analysis*, Academic Press, 1960.

[2] Struble, Raimond A., *Nonlinear Differential Equations*, McGraw-Hill, New York, 1962.

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