Student Guide for *Exploring Geometry*

Michael Hvidsten
Gustavus Adolphus College

DRAFT: August 23, 2011

Contents

Chapter 1

Geometry and the Axiomatic Method

The development of the axiomatic method of reasoning was one o the most profound events in the history of mathematics. In th chapter we explore axiomatic systems and their properties.

One strand running through the chapter is the search for th “ideal”. The golden ratio is the ideal in concrete form, realize through natural and man-made constructions. Deductive reasonin from a base set of axioms is the ideal in abstract form, realized i the crafting of clear, concise, and functional definitions, and in th reasoning employed in well-constructed proofs.

Another strand in the chapter, and which runs through the enti text, is that of the interplay between the concrete and the abstrac As you work through this text, you are encouraged to “play” wit concrete ideas, such as how the Golden Ratio appears in nature, bu you are also encouraged to play (experiment) when doing proofs an more abstract thinking. The experimentation in the latter is of th mind, but it can utilize many of the same principles of exploratio as you would use in a computer lab. When trying to come up wit a proof you should consider lots of examples and ask “What if ...? questions. Most importantly, you should *interact* with the idea just as you interact with a computer lab project.

Student interaction with ideas and discovery of concepts is primary organizing principle for the text. Interaction is encourage in three ways. First, topics are introduced and developed in th text. Next, lab projects reinforce concepts, or introduce relate ideas. Lastly, project results are discussed, and conclusions draw in written lab reports. You will first read about concepts and hea them discussed in class. Then, you will conduct ”experiments” t make the ideas concrete. Finally, you will conceptualize ideas b re-telling them in project reports.

The work you do in the lab and in group projects is a critic component of the course. The projects that are designed to be do in groups have an additional pedagogical advantage. You will fin that by speaking with other students, using mathematical terms an concepts, you will better internalize such concepts and make the less abstract.

Notes on Lab Projects

The main difficulty you will face with the first lab project will be i learning the functionality of the *Geometry Explorer* program. O major point to watch out for is the notion of “attaching” objec together when doing their construction. For example, when yo create a point on top of a line, the point becomes attached to th line. That is, when the point is moved it is constrained to follow th line.

In order to help with the formatting of lab reports there is sample lab report for a “fake” lab on the Pythagorean Theorem i appendix A of this guide.

Solutions to Exercises in Chapter 1

1.3 Project 1 - The Ratio Made of Gold

1.3.1 Since $AB = 2$, we have that $\frac{2}{x} = \frac{1+\sqrt{5}}{2}$. Solve for x and clea the denominator of radicals.

1.3.3 Have some fun with this one, but do not get carried awa with this idea and spend the whole class period on it!

1.4 The Rise of the Axiomatic Method

In this section we focus on *reasoning* in mathematics. The prob lems in this section may seem quite distant from the geometry yo learned in high school, but the goal is to practice reasoning from th definitions and properties that an axiomatic system posits and the argue using just those basic ideas and relationships. This is goo mental training. It is all too easy to argue from diagrams whe trying to justify geometric statements.

1.4.1 If dictionaries were not circular, there would need to be a infinite number of different words in the dictionary.

1.4.3 Let a set of two different flavors be called a pairing. Sup pose there were m children and $n > m$ pairings. By Axiom 2 ever pairing is associated to a unique child. Thus, for some two pairing P_1,P_2 there is a child C associated to both. But this contradic Axiom 3. Likewise, if $m > n$, then by Axiom 3 some two childre would have the same pairing. This contradicts Axiom 2. So, $m =$ and, since the number of pairings is $4 + 3 + 2 + 1 = 10$, there are 1 children.

1.4.5 There are exactly four pairings possible of a given flavo with the others. By exercise 1.4.3 we know that there are fou distinct children associated to these pairings.

1.4.7 By Axioms 2 and 4 we have $ex = (xx^{-1})x = x(x^{-1}x)$. S all we need to do is show that $x^{-1}x = e$. Now, $(x^{-1}x)(x^{-1}x)$ $x^{-1}(xx^{-1})x = x^{-1}ex = x^{-1}x$, by Axioms 2, 3, and 4. Let $y = x^{-1}$ Then, $yy = y$ and $yyy^{-1} = yy^{-1}$ by Axiom 4. So, $y = e$ by Axiom

3 and 4 and the proof is complete. Note: This proof is a bit trick – you may want to first experiment with xx^{-1}.

1.4.9 First we show that $1 \in M$. By Axiom 4 we know 1 is no the successor of any natural number. In particular, it cannot be successor of itself. Thus, $1' \neq 1$ and $1 \in M$. Now, suppose $x \in M$ That is, $x' \neq x$. By Axiom 3 we have that $(x')' \neq x'$, and so $x' \in M$ Both conditions of Axiom 6 are satisfied and thus $M = N$.

1.4.11 Given x, let $M = \{y | x + y \; is \; defined\}$. Then, by def nition $1 \in M$. Suppose $y \in M$. Then, $x + y' = (x + y)'$ is define and $y' \in M$. So, $M = N$ by Axiom 6. Now, since x was chose arbitrarily, addition is defined for all x and y.

1.4.13 This is a good discussion question. Think about the ro of abstraction versus application in mathematics. Think about ho abstraction and application cross-fertilize one another.

1.5 Properties of Axiomatic Systems

This is a "meta" section. By this is meant that we are studyin properties of axiomatic systems themselves, considering such sy tems as mathematical objects in comparison to other systems. Th may seem quite foreign territory to you, but have an open min and think about how one really knows that mathmeatics is true o logically consistent. We often think of mathematics as an ancien subject, but in this section we bring in the amazing results of th twentieth century mathematician Kurt Godel.

If this topic interests you, you may want to further researc the area of information theory and computability in computer sc ence. A good reference here is Gregory Chaitin's book *The Limi of Mathematics* (Springer, 1998.) Additionally, much more could b investigated as to the various philosophies of mathematics, in pa ticular the debates between platonists and constructionists, or be tween intuitionists and formalists. A good reference here is Edna E Kramer's *The Nature and Growth of Modern Mathematics* (Prince ton, 1981), in particular Chapter 29 on Logic and Foundations.

1.5.1 Let S be the set of all sets which are not elements o themselves. Let P be the proposition that “S is an element of itself” And consider the two propositions P and the negation of P, whic we denote as $\neg P$. Assume P is true. Then, S is an element of itsel So, S is a set which by definition is not an element of itself. So, $\neg$ is true. Likewise, if $\neg P$ is true then P is true. In any event we g P and $\neg P$ both true, and the system cannot be consistent.

1.5.3 Good research books for this question are books on th history of mathematics. This could be a good final project idea.

1.5.5 Let P be a point. Each pairing of a point with P is asso ciated to a unique line. There are exactly three such pairings.

1.5.7 Yes. The lines and points satisfy all of the axioms.

1.5.9 If (x, y) is in P, then $x < y$. Clearly, $y < x$ is impossib and the first axiom is satisfied. Also, inequality is transitive o numbers so the second axiom holds and this is a model.

1.6 Euclid’s Axiomatic Geometry

In this section we take a careful look at Euclid’s original axiomat system. We observe some of its inadequacies in light of our moder “meta” understanding of such systems, and discuss the one axio that has been the creative source of much of modern geometry – th Parallel Postulate.

1.6.1 Good research books for this question are books on th history of mathematics.

1.6.3 An explanation can be given based on a figure like th following:

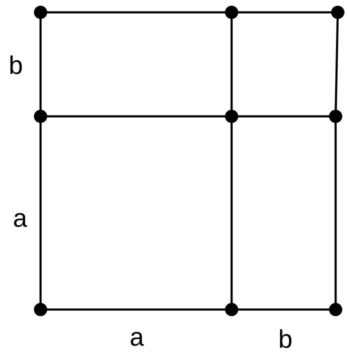

Figure 1.1:

1.6.5

$$\begin{aligned}123 &= 3 \cdot 36 + 15\\ 36 &= 2 \cdot 15 + 6\\ 15 &= 2 \cdot 6 + 3\\ 6 &= 2 \cdot 3 + 0\end{aligned}$$

Thus, $gcd(123, 36) = 3$.

1.6.7 This exercise is a good starting off point for discussing th importance of definitions in mathematics. One possible definitio for a circle is:

Definition 1.1. A circle with center O and radius length r is th set of points P on the sphere such that the distance along the grea circle from O to P is r.

Note that this definition is itself not entirely well-defined, as w have not specified what we mean by distance. Here, again, is a goo opportunity to wrestle with the "best" definition of distance. Fo circles of any radius to exist, distance must be defined so that grows without bound. Thus, one workable definition is for distanc to be net *cumulative* arclength along a great circle as we move fro a point O to a point P.

An angle ABC can be most easily defined as the Euclidean ang made by the tangent lines at B to the circles defining $\overleftrightarrow{AB}$ and $\overleftrightarrow{C}$

Then, Postulate 1 is satisfied as we can always construct a grea circle passing through two points on the sphere. If the points ar antipodal, we just use any great circle through those points. Othe wise, we simply intersect the sphere with the plane through the tw points and the center of the sphere.

Postulate 2 is satisfied as we can always extend an arc of a grea circle, although we may retrace the existing arc.

Postulate 3 is satisfied if we use the cumulative distance defin tion as discussed above.

Postulate 4 is automatically satisfied as angles are Euclidea angles.

Postulate 5 is not satisfied, as *every* pair of lines intersects. A easy proof of this is to observe that every line is uniquely defined b a plane through the origin. Two non-parallel planes will intersect i a line, and this line must intersect the sphere at two points.

1.6.9 This is true. Use a plane argument. Given a plane throug the origin, we can always find an orthogonal plane. The angle thes planes make will equal the angle of the curves they define on th sphere, as the spherical angles are defined by tangent lines to th sphere, and thus lie in the planes.

1.6.11 Yes. An example is the triangle that is defined in th first octant by intersecting the sphere with each of the three positiv coordinate axes. This triangle has three right angles.

1.7 Project 2 - A Concrete Axiomatic System

After the last few sections dealing with abstract axiomatic system this lab is designed so that you can explore another geometric syste through concrete manipulation of the points, lines, etc of that sy tem. The idea here is to have you explore the environment first, the make some conjectures about what is similar and what is differe in this system as compared to standard Euclidean geometry.

1.7.1 You should report the results of your experiments her You do not yet have the tools to prove these results, but you shoul provide evidence that you have fully explored each idea.

For example, you could report that you tried to construct a rec angle, but were unsuccessful in doing so. You may discover that you construct a four-sided figure with three right angles, the fourt angle is always less than ninety degrees.

The sum of the angles in a triangle will be less than 180 degree

Euclid's construction of an equilateral triangle is valid in hype bolic geometry. Again, you should provide experimental evidenc for this.

Finally, the perpendicular to a line through a point not on th line is a valid construction. Here, it is enough for you to experimen with the built-in perpendicular construction tool to create a ne line that *always* stays perpendicular to a given line.

Chapter 2

Euclidean Geometry

In this chapter we start off with a very brief review of basic properti
of angles, lines, and parallels.

Solutions to Exercises in Chapter 2

2.1 Angles, Lines, and Parallels

This section may be the least satisfying section in the chapter fo you, since many theorems are referenced without proof. These re sults were (hopefully) covered in great detail in your high scho geometry course and we will only briefly review them. A full an consistent development of the results in this section would enta a "filling in" of many days foundational work based on Hilbert axioms.

A significant number of the exercises in this section deal wit parallel lines. This is for two reasons. First of all, historically ther was a great effort to prove Euclid's fifth Postulate by converting into a logically equivalent statement that was hoped to be easier t prove. Thus, many of the exercises nicely echo this history. Se ondly, parallels and the parallel postulate are at the heart of on of the greatest revolutions in math—the discovery of non-Euclidea

geometry. This section foreshadows that development, which is cov
ered in Chapters 7 and 8.

2.1.1 It has already been shown that $\angle FBG \cong \angle DAB$. Also, b
the vertical angle theorem (Theorem 2.3) we have $\angle FBG \cong \angle EB$
and thus, $\angle DAB \cong \angle EBA$.

Now, $\angle DAB$ and $\angle CAB$ are supplementary, thus add to tw
right angles. Also, $\angle CAB$ and $\angle ABF$ are congruent by the firs
part of this exercise, as these angles are alternate interior angle
Thus, $\angle DAB$ and $\angle ABF$ add to two right angles.

2.1.3.a False, right angles are defined solely in terms of congru
ent angles.

2.1.3.b False, an angle is defined as *just* the two rays plus th
vertex.

2.1.3.c True. This is part of the definition.

2.1.3.d False. The term "line" is undefined.

2.1.5 Proposition I-23 states that angles can be copied. Let
and B be points on l and n respectively and let m be the line throug
A and B. If $t = m$ we are done. Otherwise, let D be a point on
that is on the same side of n as l. (Assuming the standard properti
of betweenness) Then, $\angle BAD$ is smaller than the angle at A forme
by m and n. By Theorem 2.9 we know that the interior angles at
and A sum to two right angles, so $\angle CBA$ and $\angle BAD$ sum to le
than two right angles. By Euclid's fifth postulate t and l must mee

2.1.7 First, assume Playfair's Postulate, and let lines l and
be parallel, with line t perpendicular to l at point A. If t does no
intersect m then, t and l are both parallel to m, which contradic
Playfair. Thus, t intersects m and by Theorem 2.9 t is perpendicula
at this intersection.

Now, assume that whenever a line is perpendicular to one of tw
parallel lines, it must be perpendicular to the other. Let l be a lin
and P a point not on l. Suppose that m and n are both parallel to
at P. Let t be a perpendicular from P to l. Then, t is perpendicula
to m and n at P. By Theorem 2.4 it must be that m and n ar
coincident.

2.1.9 Assume Playfair and let lines m and n be parallel to lin l. If $m \neq n$ and m and n intersect at P, then we would have tw different lines parallel to l through P, contradicting Playfair. Thu either m and n are parallel, or are the same line.

Conversely, assume that two lines parallel to the same line ar equal or themselves parallel. Let l be a line and suppose m and are parallel to l at a point P not on l. Then, n and m must be equa as they intersect at P.

2.2 Congruent Triangles and Pasch’s Axiom

This section introduces many results concerning triangles and als discusses several axiomatic issues that arose from Euclid’s treatmen of triangles.

2.2.1 Yes, it could pass through points A and B of ΔABC. does not contradict Pasch’s axiom, as the axiom stipulates that th line cannot pass through A, B, or C.

2.2.3 No. Here is a counter-example.

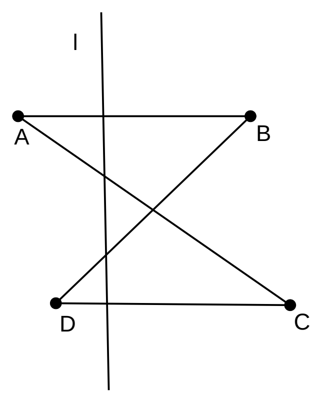

Figure 2.1:

2.2.5 If $A = C$ we are done. If A, B, and C are collinear, the B cannot be between A and C, for then we would have two poin of intersection for two lines. If A is between B and C, then l canno intersect $\overline{AC}$. Likewise, C cannot be between A and B.

If the points are not collinear, suppose A and C are on opposit sides. Then l would intersect all three sides of ΔABC, contradictin Pasch's axiom.

2.2.7 Let $\angle ABC \cong \angle ACB$ in ΔABC. Let $\overrightarrow{AD}$ be the ang bisector of $\angle BAC$ meeting side $\overline{BC}$ at D. Then, by AAS, ΔDB and ΔDCA are congruent and $\overline{AB} \cong \overline{AC}$.

2.2.9 Suppose that two sides of a triangle are not congruen Then, the angles opposite those sides cannot be congruent, as if the were, then by the previous exercise, the triangle would be isoscele

Suppose in ΔABC that $\overline{AC}$ is greater than $\overline{AB}$. On $\overline{AC}$ we ca find a point D between A and C such that $\overline{AD} \cong \overline{AB}$. Then, $\angle AD$ is an exterior angle to ΔBDC and is thus greater than $\angle DCB$. Bu ΔABD is isosceles and so $\angle ADB \cong \angle ABD$, and $\angle ABD$ is great than $\angle DCB = \angle ACB$.

2.2.11 Let ΔABC and ΔXYZ be two right triangles with righ angles at A and X, and suppose $\overline{BC} \cong \overline{YZ}$ and $\overline{AC} \cong \overline{XZ}$. Suppos $\overline{AB}$ is greater than $\overline{XY}$. Then, we can find a point D between A an B such that $\overline{AD} \cong \overline{XY}$. By SAS $\Delta ADC \cong \Delta XYZ$. Now, $\angle BDC$ exterior to ΔADC and thus must be greater than 90 degrees. Bu ΔCDB is isosceles, and thus $\angle DBC$ must also be greater than 9 degrees. This is impossible, as then ΔCDB would have angle su greater than 180 degrees.

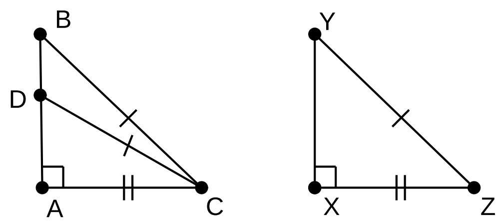

Figure 2.2:

2.3 Project 3 - Special Points of a Triangle

You are encouraged to explore and experiment in this lab projec Are there any other sets of intersecting lines that one could constru for a given triangle? Are there interesting properties of constructe intersecting lines in other polygons?

2.3.1 ΔDGB and ΔDGA are congruent by SAS, as are ΔEG and ΔEGC. Thus, $\overline{AG} \cong \overline{BG} \cong \overline{CG}$. By SSS $\Delta AFG \cong \Delta CF$ and since the angles at F must add to 180 degrees, the angles at must be congruent right angles.

2.3.3 The angle pairs in question are all pairs of an exteri angle and an interior angle on the same side for a line falling on tw parallel lines. These are congruent by Theorem 2.9.

Since $\angle DAB$, $\angle BAC$, and $\angle CAE$ sum to 180 degrees, an $\angle BDA$, $\angle BAD$, and $\angle ABD$ sum to 180 then, using the congru ences shown in the diagram, we get that $\angle DBA \cong \angle BAC$. Lik wise, $\angle BAD \cong \angle ABC$. By ASA we get that $\Delta ABC \cong \Delta BA$ Similarly, $\Delta ABC \cong \Delta CEA$ and $\Delta ABC \cong \Delta FCB$.

2.3.5 Let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ define an angle and let $\overrightarrow{AD}$ be the b sector. Drop perpendiculars from D to $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and assum these intersect at B and C. Then, by AAS, ΔABD and ΔACD a congruent, and $\overline{BD} \cong \overline{CD}$.

Conversely, suppose D is interior to $\angle BAC$ with $\overline{BD}$ perpendi

ular to $\overrightarrow{AB}$ and $\overline{CD}$ perpendicular to $\overrightarrow{AC}$. Also, suppose that $\overline{BD}$
$\overline{CD}$. Then, by the Pythagorean Theorem $AB^2 + BD^2 = AD^2$ an
$AC^2+CD^2 = AD^2$. Thus, $\overline{AB} \cong \overline{AC}$ and by SSS $\Delta ABD \cong \Delta ACI$
This implies that $\angle BAD \cong \angle CAD$.

2.4.1 Mini-Project:Area in Euclidean Geometry

This section includes the first "mini-project" for the course. Thes projects are designed to be done in the classroom, in groups of thre or four. Each group should elect a Recorder. The Recorder's so job is to outline the group's solutions to exercises. The summar should not be a formal write-up of the project, but should give brief synopsis of the group's reasoning process.

The main goal for the mini-projects is to have discussion of g ometric ideas. Through the group process, you can clarify you understanding of concepts, and help others better grasp abstrac ways of thinking. There is no better way to conceptualize an ide than to have to explain it to another person.

In this mini-project, you are asked to grapple with the notio of "area". The notion of area is not that simple or obvious. F example, what does it mean for two figures to have the same area

2.4.1 Construct a diagonal and use the fact that alternate inte rior angles of a line falling on parallel lines are congruent to generat an ASA congruence for the two sub-triangles created in the para lelogram.

2.4.3 If the figure can be split into triangle pieces that can b separated into congruent pairs, then, since triangles are polygons, can be split into congruent pairs of polygonal pieces.

On the other hand, it it can be split into congruent polygon pieces, then we can split the polygon pieces into triangles, and w can use SAS repeatedly to generate congruent pairs of triangles.

2.4.5 Use Theorem 2.8 and Exercise 2.4.1.

Project Report

Hidden Assumptions? One hidden assumption is the notion tha areas are additive. That is, if we have two figures that are no overlapping, then the area of the union is the sum of the separat areas.

2.4.2 Cevians and Area

2.4.7 Since a median is a cevian to a midpoint, then the fraction in the ratio product of Theorem 2.24 are all equal to 1.

2.4.9 Refer to the figure below. By the previous exercise w know that $1+2+3=4+5+6$ (in terms of areas). Also, since 1 an 2 share the same base and height we have $3=4$. Similarly, $1=$ and $5=6$. Thus, $1=6$.

Similarly, $2+3+4=1+5+6$ will yield $4=5$, and $3+4+5$ $1+5+5$ yields $2=3$. Thus, all 6 have the same area.

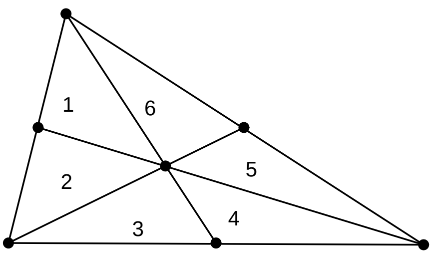

Figure 2.3:

2.5 Similar Triangles

As stated in the text, similarity is one of the most useful tools in th geometer's toolkit. It can be used in the definition of the trigono metric functions and in proofs of theorems like the Pythagorea Theorem.

2.5.1 Since $\overleftrightarrow{DE}$ cuts two sides of triangle at the midpoints, the by Theorem 2.27, this line must be parallel to the third side $\overline{BC}$ Thus $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Since the angle at

is congruent to itself, we have by AAA that ΔABC and ΔADE a
similar, with proportionality constant of $\frac{1}{2}$.

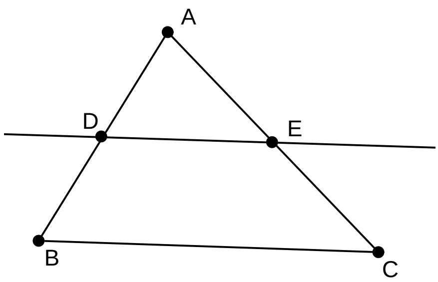

Figure 2.4:

2.5.3 Let ΔABC and ΔDEF have the desired SSS similarit property. That is sides $\overline{AB}$ and $\overline{DE}$, sides $\overline{AC}$ and $\overline{DF}$, and side $\overline{BC}$ and $\overline{EF}$ are proportional. We can assume that $\overline{AB}$ is at leas as large as $\overline{DE}$. Let G be a point on $\overline{AB}$ such that $\overline{AG} \cong \overline{DE}$. L $\overleftrightarrow{GH}$ be the parallel to $\overleftrightarrow{BC}$ through G. Then, $\overleftrightarrow{GH}$ must interse $\overleftrightarrow{AC}$, as otherwise $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ would be parallel. By the properti of parallels, $\angle AGH \cong \angle ABC$ and $\angle AHG \cong \angle ACB$. Thus, ΔAG and ΔABC are similar.

Therefore, $\frac{AB}{AG} = \frac{AC}{AH}$. Equivalently, $\frac{AB}{DE} = \frac{AC}{AH}$. We are give that $\frac{AB}{DE} = \frac{AC}{DF}$. Thus, $\overline{AH} \cong \overline{DF}$.

Also, $\frac{AB}{AG} = \frac{BC}{GH}$ and $\frac{AB}{AG} = \frac{AB}{DE} = \frac{BC}{EF}$. Thus, $GH \cong EF$.

By SSS ΔAGH and ΔDEF are congruent, and thus ΔABC a ΔDEF are similar.

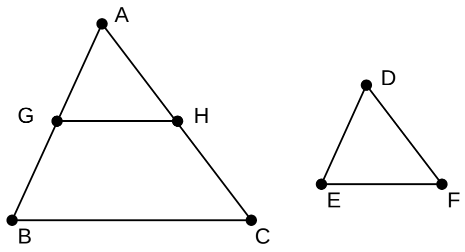

Figure 2.5:

2.5.5 Any right triangle constructed so that one angle is co[n]gruent to $\angle A$ must have congruent third angles, and thus the co[n]structed triangle must be similar to ΔABC. Since *sin* and *cos* a[re] defined in terms of ratios of sides, then proportional sides will hav[e] the same ratio, and thus it does not matter what triangle one us[e]s for the definition.

2.5.7 If the parallel to $\overleftrightarrow{AC}$ does not intersect $\overleftrightarrow{RP}$, then it woul[d] be parallel to this line, and since it is already parallel to $\overleftrightarrow{AC}$, then b[y] exercise 2.1.15 $\overleftrightarrow{RP}$ and $\overleftrightarrow{AC}$ would be parallel, which is impossible

By the properties of parallels, $\angle RAP \cong \angle RBS$ and $\angle RPA$ $\angle RSB$. Thus, by AAA ΔRBS and ΔRAP are similar. ΔPCQ an[d] ΔSBQ are similar by AAA using an analogous argument for two o[f] the angles and the vertical angles at Q.

Thus, $\frac{CP}{BS} = \frac{CQ}{BQ} = \frac{PQ}{QS}$, and $\frac{AP}{BS} = \frac{AR}{BR} = \frac{PR}{SR}$. So, $\frac{CP}{AP}\frac{BQ}{QC}$ $\frac{CP}{AP}\frac{BS}{CP} = \frac{BS}{AP}$ And, $\frac{CP}{AP}\frac{BQ}{QC}\frac{AR}{RB} = \frac{BS}{AP}\frac{AR}{RB} = \frac{BS}{AP}\frac{AP}{BS} = 1$.

2.5.1 Mini-Project: Finding Heights

This mini-project is a very practical application of the notion o[f] similarity. The mathematics in the first example for finding heig[ht] is not hard, but the interesting part is the data collection. You wi[ll] need to determine how to get the most accurate measurements usin[g] the materials on hand.

The second method of finding height is a calculation using tw similar triangles. The interesting part of this project is to see th connection between the mirror reflection and the calculation yo made in part I.

You should work in small groups with a Recorder, but make sur the Recorder position gets shifted around from project to project.

2.6 Circle Geometry

This section is an introduction to the basic geometry of the circl The properties of inscribed angles and tangents are the most impo tant properties to focus on in this section.

2.6.1 Case I: A is on the diameter through $\overline{OP}$. Let α $m\angle PBO$ and $\beta = m\angle POB$. Then, $\beta = 180 - 2\alpha$. Also, $m\angle AOB$ $180 - \beta = 2\alpha$.

Case II: A and B are on the same side of $\overleftrightarrow{PO}$. We can assum that $m\angle OPB > m\angle OPA$. Let $m\angle OPB = \alpha$ and $m\angle OPA =$ Then, we can argue in a similar fashion to the proof of the Theore using $\alpha - \beta$ instead of $\alpha + \beta$.

2.6.3 Consider $\angle AQO$ where O is the center of the circle throug A. This must be a right angle by Corollary 2.33. Similarly, $\angle BQ$ must be a right angle, where O' is the center of the circle throug B. Thus, A, Q, and B are collinear.

2.6.5 Let $\overline{AB}$ be the chord, O the center, and M the midpoi of $\overline{AB}$. Then $\Delta AOM \cong \Delta BOM$ by SSS and the result follows.

2.6.7 Consider a triangle on the diagonal of the rectangle. Th has a right angle, and thus we can construct the circle on this angl Since the other triangle in the rectangle also has a right angle o the same side (the diameter of the circle) then it is also inscribed i the same circle.

2.6.9 Suppose they intersected at another point P. Then, ΔTB and ΔTAP are both isosceles triangles. But, this would imply, b the previous exercise, that there is a triangle with two angles great than a right angle, which is impossible.

2.6.11 Let P and Q be points on the tangent, as shown. Ther $\angle BDT \cong \angle BTP$, as both are inscribed angles on the same ar Likewise, $\angle ACT \cong \angle ATQ$. Since, $\angle BTP \cong \angle ATQ$ (vertical ar gles), then $\angle BDT \cong \angle ACT$ and the lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are paralle

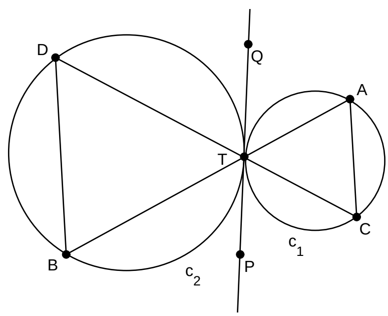

Figure 2.6:

2.6.13 Suppose that the bisector did not pass through the cente Then, construct a segment from the center to the outside point. B the previous theorem, the line continued from this segment mus bisect the angle made by the tangents. But, the bisector is uniqu and thus the original bisector must pass through the center.

2.7 Project 4 - Circle Inversion and Orthogonality

This section is crucial for the later development of the Poincar model of non-Euclidean (hyperbolic) geometry. It is also has som of the most elegant mathematical results found in the course.

2.7.1 By Theorem 2.32, $\angle Q_2P_1P_2 \cong \angle Q_2Q_1P_2$. Thus, $\angle PP_1Q_2$ $\angle PQ_1P_2$. Since triangles ΔPP_1Q_2 and ΔPQ_1P_2 share the angle a P, then they are similar. Thus, $\frac{PP_1}{PQ_1} = \frac{PQ_2}{PP_2}$, or $(PP_1)(PP_2)$ $(PQ_1)(PQ_2)$.

2.7.3 By similar triangles $\frac{OP}{OT} = \frac{OT}{OP'}$. Since $OT = r$ the resu follows.

Chapter 3

Analytic Geometry

This chapter is a very quick review of analytic geometry. In suc-
ceeding chapters, analytic methods will be utilized freely.

Solutions to Exercises in Chapter 3

3.2 Vector Geometry

3.2.1 If A is on either of the axes, then so is B and the distanc
result holds by the definition of coordinates. Otherwise, A (and B
are not on either axis. Drop perpendiculars from A and B to the
axis at P and Q. By SAS similarity, ΔAOP and ΔBOQ are simila
and thus $\angle AOP \cong \angle BOQ$, which means that A and B are on th
same line $\overleftrightarrow{AO}$, and the ratio of BO to AO is k.

3.2.3 The vector from P to Q is in the same direction (or oppo-
site direction) as the vector v. Thus, since the vector from P to Q
$\vec{Q} - \vec{P}$, we have $\vec{Q} - \vec{P} = tv$, for some real number t. In coordinate
we have $(x, y) - (a, b) = (tv_1, tv_2)$, or $(x, y) = (a, b) + t(v_1, v_2)$.

3.2.5 By exercise 3.2.3 the line through A and B can be rep-
resented by the set of points of the form $\vec{A} + t(\vec{B} - \vec{A})$. Then
$M = \frac{1}{2}(\vec{A} + \vec{B}) = \vec{A} + \frac{1}{2}(\vec{B} - \vec{A})$ is on the line through A and B, an
is between A and B. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then th

distance from A to M is $\sqrt{(\frac{x_1}{2}-\frac{x_2}{2})^2+(\frac{y_1}{2}-\frac{y_2}{2})^2}$, which is equa to the distance from B to M.

3.3 Project 5 - Bézier Curves

3.3.1 The derivative to $\vec{c}(t)$ is $\vec{c}'(t) = 2\vec{B} - 2\vec{A} + 2t(\vec{C} - 2\vec{B} + \vec{A}$ Then $\vec{c}'(0) = 2\vec{B} - 2\vec{A}$, which is in the direction of $\vec{B} - \vec{A}$ an $\vec{c}'(1) = 2\vec{B}-2\vec{A}+2(\vec{C}-2\vec{B}+\vec{A} = 2\vec{C}-2\vec{B}$, which is in the directio of $\vec{C} - \vec{B}$.

3.3.3 Similar computation to Exercise 3.3.1

3.4 Angles in Coordinate Geometry

3.4.1 Let $\vec{A} = (\cos(\alpha), \sin(\alpha))$ and $\vec{B} = (\cos(\beta), \sin(\beta))$. Then, fro Theorem 3.11 we have $\cos(\alpha - \beta) = \vec{A} \circ \vec{B}$, since $\vec{A}$ and $\vec{B}$ are un length vectors. The result follows immediately.

3.4.3 By exercise 3.4.1,

$$\begin{aligned}\cos(\frac{\pi}{2} - (\alpha+\beta)) &= \cos(\frac{pi}{2})\cos(\alpha+\beta) + \sin(\frac{pi}{2}90)\sin(\alpha+\beta))\\ &= \sin(\alpha+\beta).\end{aligned}$$

Then, use the formula from Exercise 3.4.2 with the term inside co being $(\frac{pi}{2} - \alpha) + (-\beta)$.

3.5 The Complex Plane

3.5.1

$$\begin{aligned}e^{i\theta}e^{i\phi} &= (\cos(\theta) + i\ \sin(\theta))(\cos(\phi) + i\ \sin(\phi))\\ &= (\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)) + i(\cos(\theta)\sin(\phi) + \sin(\theta)\\ &= \cos(\theta+\phi) + i\ \sin(\theta+\phi)\\ &= e^{i(\theta+\phi)}\end{aligned}$$

3.5.3 Let $z = e^{i\theta}$ and $w = e^{i\phi}$ and use Exercise 3.4.1.

3.5.5 The rationalized complex numbers have the form $i\frac{-1}{2}$, and $\frac{1}{10} - i\frac{1}{5}$.

3.5.7 The line through N,P', and P can be expressed as N $t(P' - N) = (0, 0, 1) + t(X, Y, Z - 1) = (tX, tY, 1 + t(Z - 1))$. A this is a point in the $x - y$ plane, we have that. $1 + t(Z - 1) = 0$, $t = \frac{1}{1-Z}$. Thus, $\pi(P') = t(X, Y) = \frac{1}{1-Z}(X, Y)$.

3.5.9 Let $z = (x, y)$ be a point in the complex plane. Then $(X, Y, Z) = (\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1})$ will get mapped to z from the wor done in exercises 3.4.5 and 3.4.6.

3.5.11 Since $|\overline{z} - \overline{z_0}| = |z - z_0|$, the function $f(z) = \overline{z}$ ha the local scale-preserving property. Consider two curves c_1 and intersecting at z_0, parameterized so that $c_1(0) = c_2(0) = z_0$. Then the angle between their tangents is the argument of $c_1'(0)$ minu the argument of $c_2'(0)$. Under conjugation, the arguments becom negative, and thus, the difference in the angles between the conjugat curves becomes negative.

3.6 Birkhoff's Axiomatic System for Analytic Geometr

3.6.1 First, if A is associated to $x_A = t_A\sqrt{dx^2 + dy^2}$, where A $(x, y) = (x_0, y_0) + t_A(dx, dy)$, and B is associated to x_B in a simila fashion, then $|x_A - x_B| = |t_A - t_B|\sqrt{dx^2 + dy^2}$. On the other hand

$$d(A, B) = \sqrt{(t_A dx - t_B dx)^2 + (t_A dy - t_B dy)^2} = \sqrt{dx^2 + dy^2}|t_A -$$

3.6.3 Given a point O as the vertex of the angle, set O as th origin of the coordinate system. Then, identify a ray $\overrightarrow{OA}$ associate to the angle θ, with $A = (x, y)$. Let $a = ||\vec{A}|| = \sqrt{x^2 + y^2}$. Then $\sin^2(\theta) + \cos^2(\theta) = (\frac{x}{a})^2 + (\frac{y}{a})^2 = \frac{x^2+y^2}{a^2} = 1$.

3.6.5 Discussion question. One idea is that analytic geometr allows one to study geometric figures by the equations that defin them. Thus, geometry can be reduced to the arithmetic (algebra of equations.

Chapter 4

Constructions

In this chapter we cover some of the basic Euclidean constructio
and also have a lot of fun with lab projects. The origami proje
should be especially interesting, as it is an axiomatic system wi
which you can *physically* interact and explore.

The third section on constructibility may be a bit heavy and a
stract, but the relationship between geometric constructibility a
algebra is a fascinating one, especially if you have had some exp
sure to abstract algebra. Also, any mathematically literate pers
should know what the three classical construction problems are, a
how the pursuit of solutions to these problems has had a profou
influence on the development of modern mathematics.

Solutions to Exercises in Chapter 4

4.1 Euclidean Constructions

4.1.1 Use SSS triangle congruence on ΔABF and ΔDGH.

4.1.3 Use the SSS triangle congruence theorem on ΔADE a
ΔABE to show that $\angle EAB \cong \angle BAE$.

4.1.5 Use the fact that both circles have the same radius.

4.1.7 Let the given line be l and let P be the point not on

Construct the perpendicular m to l through P. At a point Q o m, but not on l, construct the perpendicular n to m. Theorem 2 implies that l and n are parallel.

4.1.9 On $\overrightarrow{BA}$ construct A' such that $BA' = a$. On $\overrightarrow{BC}$ con struct C' such that $BC' = b$. Then, SAS congruence gives $\Delta AB'C$ congruent to any other triangle with the specified data.

4.2 Project 6 - Euclidean Eggs

4.2.1 The tangent to one of the circles will meet $\overleftrightarrow{AB}$ at C at righ angles by Theorem 2.36. The tangent to the other circle will als meet $\overleftrightarrow{AB}$ at C at a right angle. Since the perpendicular to $\overleftrightarrow{AB}$ a C is unique, the tangents coincide.

4.2.3 The construction steps are implied by the figure.

4.3 Constructibility

4.3.1 Just compute the formula for the intersection.

4.3.3 Reverse the roles of the product construction.

4.3.5 For $\sqrt{3}$, use a right triangle with hypotenuse 2 and on side 1. For $\sqrt{5}$, use a right triangle with sides of length 1 and 2.

4.3.7 Consider $\frac{a}{\pi}$. This is less than a.

4.3.9 If a circle of radius r and center (x, y) has x not con structible, then $(x, y+r)$ and $(x, y-r)$ are non-constructible on th circle. We can use the same reasoning if y is not constructible. If th center is constructible, then the previous exercise gives at least tw non-constructible points for a circle of radius r whose center is at th origin. Add (x, y) to these two points to get two non-constructib points on the original circle.

4.4 Mini-Project: Origami Construction

For this project, one will need a good supply of square paper. Com mercial origami paper is quite expensive. Equally as good paper ca be made by taking notepads and cutting them into squares using paper-cutter. (Cutting works best a few sheets at a time)

4.4.1 Given $\overline{AB}$, we can fold A onto B by axiom O2. Let be the fold line of reflection created, and let l intersect AB at C Then, since the fold preserves length, we have that $AC = CB$, an $\angle ACE \cong \angle ECB$, as show in Fig. 4.1. The result follows.

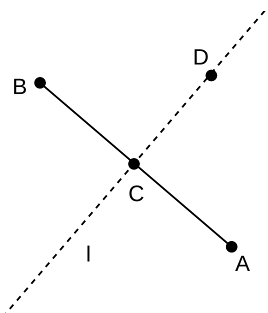

Figure 4.1:

4.4.3 Since the reflection fold across t preserves length, we hav $PR = P'R$. Also, the distance from a point to a line is measure along the perpendicular from the point to the line. Thus, the di tance from R to l is equal to $P'R$. Thus, the distance from R to equals the distance from R to l and R is on the parabola with focu P and directrix l.

An interesting result related to this construction would be t show that t is *tangent* to the parabola at R. One proof is as follow

Suppose t intersected at another point R' on the parabola. Ther by definition, R' must have been constructed in the same way tha R was, so there must be a folding (reflection) across t taking P t some point P'' on l such that $\overleftrightarrow{P''R'}$ is perpendicular to l at P'', an intersects t at R'. Then, by a triangle argument, we can show tha $\overleftrightarrow{PP'}$ and $\overleftrightarrow{PP''}$ must both be perpendicular to t at R and R'. Sinc perpendiculars are unique, we must have that $R = R'$.

(To show, for example, that $\overleftrightarrow{PP'}$ is perpendicular to t at R, w can easily show that $\Delta PQR \cong \Delta P'QR$ by using the angle- an distance-preserving properties of reflections, and then use a secon congruent triangle argument to show that $\overleftrightarrow{PP'}$ crosses t at righ angles.)

Chapter 5

Transformational Geometry

In this chapter we make great use of functional notation and some-what abstract notions such as $1-1$ and onto, inverses, compositions, etc. You may wonder how such computations are related to ge-ometry, but that is the very essence of the chapter—that we can understand and investigate geometric ideas with more than one set of mathematical techniques.

With that in mind, we will make use of synthetic geometric tech-niques where they are most elegant and can aid intuition, and at other times we will rely on analytical techniques.

Solutions to Exercises in Chapter 5

5.1 Euclidean Isometries

5.1.1 Define the function f^{-1} by $f^{-1}(y) = x$ if and only if $f(x) = y$. Then, f^{-1} is well-defined, as suppose $f(x_1) = f(x_2) = y$. Then since f is $1-1$ we have that $x_1 = x_2$. Since f is onto, we have that for every y in S there is an x such that $f(x) = y$. Thus f^{-1} is defined on all of S. Finally, $f^{-1}(f(x)) = f^{-1}(y) = x$ an

$f(f^{-1}(y)) = f(x) = y$. So, $f \circ f^{-1} = f^{-1} \circ f = id_S$.

Suppose g was another function on S such that $f \circ g = g \circ f = id$
Then, $g \circ f \circ f^{-1} = f^{-1}$, or $g = f^{-1}$.

5.1.3 Since $g^{-1} \circ f^{-1} \circ f \circ g = g^{-1} \circ g = id$ and $f \circ g \circ g^{-1} \circ f^{-1}$
$f \circ f^{-1} = id$, then $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

5.1.5 Let T be an isometry and let c be a circle centered at O
radius $r = OA$. Let $O' = T(O)$ and $A' = T(A)$. Let P be any poi
on c. Then, $O'T(P) = T(O)T(P) = OP = r$. Thus, the image of
under T is contained in the circle centered at O' of radius r. Let
be any other point on the circle centered at O' of radius r. Ther
$OT^{-1}(P') = T^{-1}(O')T^{-1}(P') = O'P' = r$. Thus, $T^{-1}(P')$ is a poi
on c and every such point P' is the image of a point on c, under th
map T.

5.1.7 Label the vertices of the triangle A, B, and C. Ther
consider vertex A. Under an isometry, consider the actual positio
of A in the plane. After applying the isometry, A might remai
or be replaced by one of the other two vertices. Thus, there a
three possibilities for the position occupied by A. Once that verte
has been identified, consider position B. There are now just tw
remaining vertices to be placed in this position. Thus, there are
maximum of 6 isometries. We can find 6 by considering the thre
basic rotations by 0, 120, and 240 degrees, and the three reflection
about perpendicular bisectors of the sides.

5.1.9 First, we show that T is a transformation. To show it
$1-1$, suppose $T(x, y) = T(x', y')$. Then, $kx + a = kx' + a$ an
$ky + b = ky' + b$. So, $x = x'$ and $y = y'$.

To show it is onto, let (x', y') be a point. Then, $T(\frac{x'-a}{k}, \frac{y'-b}{k})$
(x', y').

T is not, in general, an isometry, since if $A = (x, y)$ and B
(x', y') then $T(A)T(B) = kAB$.

5.1.11 Let ABC be a triangle and let $A'B'C'$ be its image und
T. By the previous exercise, these two triangles are similar. Thu
there is a $k > 0$ such that $A'B' = kAB$, $B'C' = kBC$, and $A'C'$
kAC. Let D be any other point not on $\overleftrightarrow{AB}$. Then, using triangl

ABD and $A'B'D'$ we get that $A'D' = kAD$.

Now, let $\overline{DE}$ be any segment with D not on $\overleftrightarrow{AB}$. Then, usin triangles ADE and $A'D'E'$ we get $D'E' = kDE$, since we know tha $A'D' = kAD$.

Finally, let $\overline{EF}$ be a segment entirely on $\overleftrightarrow{AB}$, and let D be point off $\overleftrightarrow{AB}$. Then, using triangles DEF and $D'E'F'$ we get $E'F'$ kEF, since we know that $D'E' = kDE$.

Thus, in all cases, we get that $T(A)T(B) = kAB$.

5.2.1 Mini-Project:Isometries Through Reflection

In this mini-project, you will be led through a guided discovery the amazing fact that, given any two congruent triangles, one ca find a sequence of at most three reflections taking one triangle the other.

5.2.1 First of all, suppose that C and R are on the same side $\overleftrightarrow{AB}$. Then, since there is a unique angle with side $\overline{AB}$ and measur equal to the measure of $\angle BAC$, then R must lie on $\overrightarrow{AC}$. Likewis R must lie on $\overrightarrow{BC}$. But, the only point common to these two ray is C. Thus, $R = C$.

If C and R are on different sides of $\overleftrightarrow{AB}$, then drop a perpendic ular from C to $\overleftrightarrow{AB}$, intersecting at P. By SAS, ΔPAC and ΔPA are congruent, and thus $\angle APR$ must be a right angle, and R is th reflection of C across $\overleftrightarrow{AB}$.

5.2.3 If two triangles (ΔABC and ΔPQR) share no point i common, then by Theorem 5.6 there is a reflection mapping A C, and by the previous exercise, we would need at most two mor reflections to map $\Delta r(A)r(B)r(C)$ to ΔPQR.

5.2.2 Reflections

5.2.5 Many example from nature have bilateral symmetry.

5.2.7 Let G be the midpoint of $\overline{AB}$. Then $\Delta AED \cong \Delta BCD$ b SAS and $\Delta AGD \cong \Delta BGD$ by SSS. Thus, $\overleftrightarrow{DG}$ is the perpendicula

bisector of $\overline{AB}$, and reflection across $\overleftrightarrow{DG}$ takes A to B. Also, $\overleftrightarrow{D}$
must bisect the angle at D and by the previous exercise the bisect
is a line of reflection. This proof would be easily extendable t
regular n-gons, for n odd, by using repeated triangle congruences t
show the perpendicular bisector is the angle bisector of the opposi
vertex.

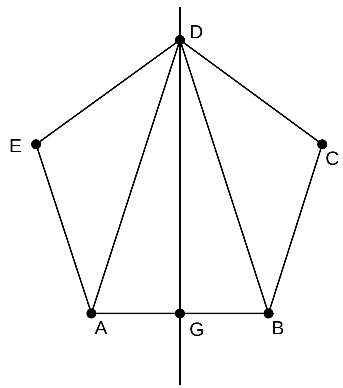

Figure 5.1:

5.2.9 Suppose that a line of symmetry l for parallelogram ABC
is parallel to side $\overline{AB}$. Then, clearly reflection across l cannot ma
A to B, as this would imply that l is the perpendicular bisector
$\overline{AB}$.

If reflection mapped A to C, then l would be the perpendicul
bisector of a diagonal of the parallelogram. But, since l is parall
to $\overline{AB}$, this would imply that the diagonal must be perpendicular t
$\overline{AB}$ as well. A similar argument can be used to show that the oth
diagonal ($\overline{BD}$) must also be perpendicular to $\overline{AB}$. If this were th
case, one of the triangles formed by the diagonals would have ang
sum greater than 180 degrees, which is impossible.

Thus, reflection across l must map A to D, and l must be th
perpendicular bisector of $\overline{AD}$. Clearly, using the property of pa
allels, we get that the angles at A and D in the parallelogram a
right angles.

5.2.11 Let r be a reflection across $\overleftrightarrow{AB}$ and let C be a poi
not on $\overleftrightarrow{AB}$. Then, $r(C)$ is the unique point on the perpendicul
dropped to $\overleftrightarrow{AB}$ at a point P on this line such that $CP = r(C)$
with $r(C) \neq C$. Now, $r(r(C))$ is the unique point on this sam
perpendicular such that $r(C)P = r(r(C))P$, with $r(r(C)) \neq r(C$
But since $r(C)P = CP$ and $C \neq r(C)$, then $r(r(C)) = C$. But, the
$r \circ r$ fixes three non-collinear points A, B, and C, and so must b
the identity.

5.2.13 Let A and B be distinct points on l. Then, $r_m \circ r_l$
$r_m(r_m(A)) = r_m(r_l(A)) = r_m(A)$ and likewise, $r_m \circ r_l \circ r_m(r_m(B))$
$r_m(B)$. Thus, the line l' through $r_m(A)$ and $r_m(B)$ is fixed by r_m
$r_l \circ r_m$ and this triple composition must be equivalent to reflecti
across l'.

5.2.15 Drop a perpendicular from O to the line intersecting a
Q. By SAS we get the length from O to P is the same as the lengt
from O' to P. Thus, to minimize the total length to V we ju
minimize the length from O' to P to V. But, the shortest path w
be a straight line, so P must be located so that it is on the li
through O' and V. Using congruent triangles and vertical angle
we see that the shortest path occurs when the two angles made a
P are congruent.

5.3 Translations

5.3.1 There are few examples in nature that have perfect tran
lational symmetry. One example might be the atoms in a cryst
atomic lattice. But there are some partial examples, like the legs o
a millipede.

5.3.3 Since $(r_2 \circ r_1) \circ (r_1 \circ r_2) = id$, and $(r_1 \circ r_2) \circ (r_2 \circ r_1) = i$
then $r_2 \circ r_1$ is the inverse of $r_1 \circ r_2$. Also, if T has translation vect
v, then $T(x, y) = (x, y) + v$. Let S be the translation defined b
$S(x, y) = (x, y) - v$. Then, $S \circ T(x, y) = ((x, y) + v) - v = (x, y)$ an
$T \circ S((x, y) - v) + v = (x, y)$. Thus, S is the inverse to T.

5.3.5 Let T_1 have translation vector v_1 and T_2 have translatio

vector v_2. Then, $T_1 \circ T_2(x, y) = T_1((x, y) + v_2) = (x, y) + (v_2 + v_1$
which is the same as$T_1 \circ T_2(x, y)$.

5.3.7 Let (x, K) be a point on the line $y = K$. If T is a tran lation with translation vector $v = (0, -K)$, then, by exercise 5.3. T^{-1} has translation vector of $-v = (0, K)$. Thus, $T^{-1} \circ r_x \circ T(x, K)$ $T^{-1} \circ r_x(x, 0) = T^{-1}(x, 0) = (x, K)$. So, $T^{-1} \circ r_x \circ T$ fixes the lin $y = K$ and so must be the reflection across this line. The coordinat equation for r is given by $T^{-1} \circ r_x \circ T(x, y) = T^{-1} \circ r_x(x, y - K)$ $T^{-1}(x, -y + K) = (x, -y + 2K)$. So, $r(x, y) = (x, -y + 2K)$.

5.3.9 Let T be a translation with (non-zero) translation vecto parallel to a line l. Let m be perpendicular to l at point P. L n be the perpendicular bisector of $\overline{PT(P)}$, intersecting $\overline{PT(P)}$ a point Q. Then, r_n, reflection about n maps P to $T(P)$. Conside $r_n \circ T$. We have $r_n \circ T(P) = P$. Let $R \neq P$ be another poi on m. Then, $PRT(R)T(P)$ is a parallelogram, and thus $\angle PRT($ and $\angle RT(R)T(P)$ are right angles. Let S be the point where intersects $\overline{RT(R)}$. Then, $\angle RSQ$ is also a right angle. Also, by congruent triangle argument, we have $\overline{RS} \cong \overline{ST(R)}$, and so n is th perpendicular bisector of $\overline{RT(R)}$ and $r_n \circ T(R) = R$. Since $r_n \circ$ fixes two points on m we have $r_n \circ T = r_m$, or $T = r_n \circ r_m$.

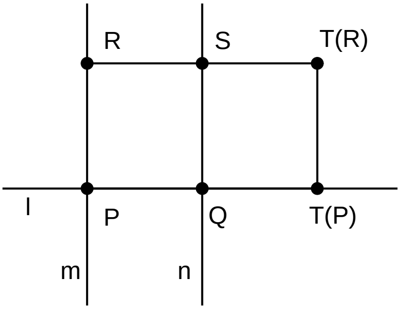

Figure 5.2:

5.4 Rotations

5.4.1 First,

$$
\begin{aligned}
T^{-1} \circ Rot_\phi \circ T(C) &= T^{-1} \circ Rot_\phi \circ T(x, y) \\
&= T^{-1} \circ Rot_\phi(0, 0) \\
&= T^{-1}(0, 0) \\
&= (x, y) \\
&= C
\end{aligned}
$$

Suppose $T^{-1} \circ Rot_\phi \circ T$ fixed another point P. Then, Rot_ϕ $T(P) = T(P)$, which implies that $T(P) = (0, 0)$, or $P = T^{-1}(0, 0)$ $(x, y) = C$. Thus, $T^{-1} \circ Rot_\phi \circ T$ must be a rotation. What is th angle for this rotation? Consider a line l through C that is parall to the x-axis. Then, T will map l to the $x - axis$ and Rot_ϕ will ma the x-axis to a line m making an angle of ϕ with the x-axis. Ther T^{-1} will preserve this angle, mapping m to a line making an ang of ϕ with l. Thus, the rotation angle for $T^{-1} \circ Rot_\phi \circ T$ is ϕ.

5.4.3 A book on flowers or diatoms (algae) would be a goo place to start.

5.4.5 By the preceding exercise, the invariant line must pas through the center of rotation. Let A be a point on the invariar line. Then, $R_O(A)$ lies on $\overleftrightarrow{OA}$ and $\overline{OA} \cong \overline{OR_O(A)}$. Either A an $R_O(A)$ are on the same side of O or are on opposite sides. If they ar on the same side, then $A = R_O(A)$, and the rotation is the identit which is ruled out. If they are on opposite sides, then the rotatio is 180 degrees. If the rotation is 180 degrees, then for every poi $A \neq O$ we have that A, O, and $R_O(A)$ are collinear, which mea that the line $\overleftrightarrow{OA}$ is invariant.

5.4.7 Let $R = r_l \circ r_m$ be a rotation about the point P whe l and m intersect. Then, since $(r_l \circ r_m) \circ (r_m \circ r_l) = id$ and $(r_m$ $r_l) \circ (r_l \circ r_m) = id$, then $R^{-1} = r_m \circ r_l$, and the angle of rotatio is the same, but in reverse direction, as the angle is twice the ang between the lines of reflection.

5.4.9 Consider $R^{-1} \circ R'$. This map fixes O and A and thus fixe $\overleftrightarrow{OA}$. So, either $R^{-1} \circ R'$ is a reflection or it is the identity. Sinc the composition of two rotations about a common point is again rotation (by the preceding exercise), then $R^{-1} \circ R' = id$ and th result follows.

5.4.11 The hint is over-kill. H is clearly a rotation, by th definition of rotations. The angle of rotation is twice the angle mad by the lines of reflection, or twice a right angle, or 180.

5.4.13 Note that $T \circ H_A \circ T^{-1}$ maps $T(A)$ back to itself. this map fixes any other point P, then $H_A \circ T^{-1}(P) = T^{-1}(P)$, an so $T^{-1}(P) = A$ or $P = T(A)$. Thus, $T \circ H_A \circ T^{-1}$ is a rotatio about $T(A)$. Then, any line through $T(A)$ will get mapped to a lin through A by T^{-1}. Then H_A will map this new line to itself, an T will map this half-turned line back to the original line. Thus, b exercise 5.4.5, $T \circ H_A \circ T^{-1}$ is a half-turn about $T(A)$.

5.5 Project 7 -Quilts and Transformations

This project is another great opportunity for the future teachers i the class to develop similar projects for use in their own teachin One idea to incorporate into a high school version of the project to bring into the class the cultural and historical aspects of quiltin

5.5.1 In your Project Report give a report of how you did th construction.

5.5.3 For bilateral symmetry, any reflection line must pass throu the center of the quilt pattern. The only patterns which have suc symmetry are: 25-Patch Star (horizontal, vertical, 45 degree, an -45 degree lines of symmetry) and Flower Basket (45 degree line symmetry).

Star Puzzle, Dutch Man's Puzzle, and 25-Patch Star all hav rotational symmetry of 90 (and thus 180 and 270) degrees.

Thus, 25-Patch Star is the only pattern with both rotational an bilateral symmetry.

5.6 Glide Reflections

5.6.1 As with translations, it will be hard to find a perfect examp of a glide symmetry in nature. But, the are many plants whos branches alternate in a glide fashion.

5.6.3 Suppose m is invariant. Then, the glide reflection ca be written as $G = T_{AB} \circ r_l = r_l \circ T_{AB}$. If $G(G(m)) = m$, the $(T_{AB} \circ r_l) \circ (r_l \circ T_{AB})(m) = T_{2AB}(m) = m$. So, m must be parall or equal to l, if it is invariant under T_{2AB}. Suppose m is parall to l. Then, $T_{AB}(m) = m$. So, $G(m) = r_l \circ T_{AB}(m) = r_l(m)$. Bu reflection of a line m that is parallel to l cannot be equal to m. Thu the only line invariant under the glide reflection is l itself.

5.6.5 The glide reflection can be written as $G = T_{AB} \circ r_l$ $r_l \circ T_{AB}$. So, $G \circ G = (T_{AB} \circ r_l) \circ (r_l \circ T_{AB}) = T_{2AB}$.

5.6.7 The set does not include the identity element.

5.6.9 The identity (rotation angle of 0) is in the set. The con position of two rotations about the same point is again a rotation b exercise 5.4.8. The inverse to a rotation is another rotation abou the same point by exercise 5.4.7. Since rotations are functions, a sociativity is automatic.

5.6.11 A discussion and diagram would suffice for this exercis

5.6.13 By using the result in Exercise 5.2.14 repeatedly, we ca reduce any even (non-identity) isometry to the product of two re flections. Also, the identity can be written as the product of tw reflections, the product of a reflection with itself. An odd isometr can be reduced to the product of three or one reflections. Sinc rotations and translations cannot be equivalent transformations t reflections and glide reflections, then an isometry cannot be bot even and odd.

5.7 Structure and Representation of Isometries

This section is a somewhat abstract digression into ways of rep resenting transformations and of understanding their structure a algebraic elements of a group. An important theme of the section

the usefulness of the matrix form of an isometry, both from a the
oretical viewpoint (classification), as well as a practical viewpoi
(animation in computer graphics).

Matrix methods (and thus transformations) are used heavily i
the field of computer animation. There are many excellent textboo
in computer graphics that one could use as reference for this purpos
For example, the book by F.S. Hill listed in the bibliography of th
text is a very accessible introduction to the subject.

5.7.1 Let $G_1 = T_{v_1} \circ r_{l_1}$ and $G_2 = T_{v_2} \circ r_{l_2}$ be two glide reflection
If $G_1 \circ G_2$ is a translation, say T_v, then, since $G_1 \circ G_2 = T_v$
$(T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2})$, then $T_{v-v_1-v_2} = r_{l_1} \circ r_{l_2}$ and thus $l_1 || l_2$.

On the other hand, if the lines are parallel, then $G_1 \circ G_2$
$(T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ T_v \circ T_{v_2}$, for some vector v.

If the lines intersect, then the composition of r_{l_1} with r_{l_2} will b
a rotation, say R, and $G_1 \circ G_2 = (T_{v_1} \circ r_{l_1}) \circ (r_{l_2} \circ T_{v_2}) = T_{v_1} \circ R \circ T_v$
This last composition yields a rotation, by Theorem 5.20.

5.7.3 First, $f \circ r_m \circ f^{-1}(f(m)) = f(m)$, so $f(m)$ is a fixed li
for $f \circ r_m \circ f^{-1}$. Also, $(f \circ r_m \circ f^{-1})^2 = f \circ r_m \circ f^{-1} \circ f \circ r_m \circ f^{-1} = i$
Thus, $f \circ r_m \circ f^{-1}$, which must be a reflection or glide reflection fro
looking at Table 5.3, is a reflection. Since it fixes $f(m)$ it must b
reflection across $f(m)$.

5.7.5 Using the previous exercises we have $f \circ r_m \circ T_{AB} \circ f^{-1}$
$f \circ r_m \circ f^{-1} \circ f \circ T_{AB} \circ f^{-1} = r_{f(m)} \circ T_{f(A)f(B)}$.

5.7.7 Rotation of (x, y) by an angle ϕ yields $(x\ cos(\phi) - y\ sin(\phi)$
$y\ cos(\phi))$. Multiplying $x + iy$ by $cos(\phi) + i\ sin(\phi)$ yields the sam
point. Translation by $v = (v_1, v_2)$ yields $(x + v_1, y + v_2)$. Addi
$v_1 + iv_2$ to $x + iy$ yields the same result. Finally, reflection across
is given by $r_x(x, y) = (x, -y)$. Complex conjugation sends $x + iy$
$x - iy$. Clearly, this has the same effect.

5.7.9 $T_v \circ R_\beta(z) = (e^{i\beta} z) + v$. To find the fixed point set $(e^{i\beta} z)$
$v = z$ and solve for z.

5.8 Project 8 - Constructing Compositions

The purpose of this lab is to make concrete the somewhat abstrac notion of composition of isometries. In particular, by carrying ou the constructions of the lab, you will see how the conditions o compositions of rotations found in Table 5.3 arise naturally.

If you have difficulty getting started with the first proof, thin about how we can write a rotation as the composition of two re flections through the center of rotation. Note that the choice o reflection lines is not important – one can choose *any* two lines a long as they make the right angle, namely half the desired rotatio angle.

5.8.1 A rotation can be expressed as the composition of tw reflections about lines through the center of rotation, as long as th reflection lines make an angle of half the reflection angle. Since m and n are bisectors of the rotation angles, then, $R_{A,\angle EAB} = r_n \circ r_{\overleftrightarrow{AB}}$ and $R_{B,\angle ABE} = r_{\overleftrightarrow{AB}} \circ r_m$, taking into the account the orientatio of the rotation angles.

5.8.3 The rotation angle γ is twice the angle at O in ΔAOB This angle is $\angle BOA$. (Note - positively oriented) Then, taking car to measure orientation correctly, we have

$$\begin{aligned} \gamma &= 2(180 - (\angle BAO + \angle OBA) \\ &= 360 + (2\angle OAB + 2\angle ABO) \\ &= 360 + (\angle EAB + \angle ABE) \end{aligned}$$

Thus, $\gamma = (\angle EAB + \angle ABE)\ (mod 360)$.

Chapter 6

Symmetry

This chapter is quite algebraic in nature—focusing on the differen
discrete symmetry groups that arise for frieze patterns and wallpape
patterns.

Solutions to Exercises in Chapter 6

6.1 Finite Plane Symmetry Groups

6.1.1 Flowers and diatoms make good examples.

6.1.2 The symmetry group is the dihedral group of order 4
(4 rotations generated by a rotation of 90 degrees, and reflection
generated by a reflection across a perpendicular bisector of a side
This gives 8 symmetries. There are no more, since if we label th
vertices and fix a position for a vertex to occupy, we have 4 choice
for the vertex to be placed in that position and only two choice
for the rest of the vertices. Thus, a maximum of eight symmetrie
possible.

6.1.3 The dihedral group of order 5. (5 rotations generated b
a rotation of 72 degrees, and reflections generated by a reflectio
across a perpendicular bisector of a side) This gives 10 symmetrie
There are no more, since if we label the vertices and fix a positio

for a vertex to occupy, we have 5 choices for the vertex to be place in that position and only two choices for the rest of the vertice Thus, a maximum of ten symmetries possible.

6.1.5 Using an argument like that used in exercises 6.1.2 an 6.1.3, we know there are at most $2n$ symmetries. Also, by the wor done in section 5.4 we know there are n rotations, generated by rotation of $\frac{360}{n}$, that will be symmetries. Let r be a reflection acros a perpendicular bisector of a side. This will be a reflection, as wi all n compositions of this reflection with the n rotations. This give $2n$ different symmetries.

6.1.7 The number of symmetries is $2n$. The only symmetri that fix a side are the identity and a reflection across the perpen dicular bisector of that side. The side can move to n different side Thus, the stated product is $2n$ as claimed.

6.2 Frieze Groups

6.2.1 Since $\gamma^2 = \tau$, then $< \tau, \gamma, H >$ is contained in $< \gamma, H$ Also, it is clear that $< \gamma, H >$ is contained in $< \tau, \gamma, H >$. Thu $< \tau, \gamma, H >=< \gamma, H >$.

6.2.3 Let r_u and $r_{u'}$ be two reflections across lines perpendicul to m. Then, the composition $r_u \circ r_{u'}$ must be a translation, as thes lines will be parallel. Thus, $r_u \circ r_{u'} = T^k$ for some k, and $r_{u'} = r_u \circ T$

6.2.5 Consider g^2. This must be a translation, so $g^2 = T_{kv}$ f some k where T_v is the fundamental translation. Then, $g = T_{\frac{k}{2}v} \circ r_r$ where m is the midline. Suppose $\frac{k}{2}$ is an integer, say $\frac{k}{2} = j$. Then since $T_{(v-jv)}$ is in the group, we have $T_{(v-jv)} \circ g = T_{(v-jv)} \circ T_{\frac{k}{2}v} \circ r_m$ $T_v \circ r_m$ is in the group.

Otherwise, $\frac{k}{2} = j + \frac{1}{2}$ for some integer j. We can find T_{-jv} the group such that $T_{-jv} \circ g = T_{\frac{v}{2}} \circ r_m$ is in the group.

6.2.7 The composition $r_v \circ r_u$ must be a translation. Also, $r_v \circ r_u(A) = r_v(A) = C$, then the translation vector must be $\vec{AC}$ But, the length of $\overline{AC}$ is twice that of $\overline{AB}$. So, we get that $2\vec{AB}$ $k'v$ for some k'. Now, either k' is even or it is odd. The resu follows.

6.2.9 From Table 4.1 we know that $\tau \circ H$ or $H \circ \tau$ is either translation or a rotation, so it must be either τ^k for some k or H for A on m. Thus, any composition of products of τ and H can b reduced ultimately to a simple translation or half-turn, or to som $\tau^j \circ H_B$ or $H_B \circ \tau_j$, which are both half-turns. Thus, the subgrou generated by τ and H cannot contain r_m or r_u or γ and none $< \tau, r_m >$ or $< \tau, r_u >$ or $< \tau, r_m >$ can be subgroups of $< \tau, H >$

6.2.11 The compositions $\tau^k \circ r_m$ or $r_m \circ \tau^k$ generate glide r flections with glide vectors kv. The composition of τ with suc glide reflections generates other glide reflections with glide vecto $(k + j)v$. The composition of r_m with a glide in the direction m will generate a translation. Thus, compositions of the thre types of symmetries—glides, r_m, and τ^k—will only generate sym metries within those types. Thus, $< \tau, \gamma >$ cannot be a subgrou of $< \tau, r_m >$, since γ has translation vector of $\frac{v}{2}$ which cannot b generated in $< \tau, r_m >$. Also, neither $< \tau, r_u >$ nor $< \tau, H >$ ca be subgroups of $< \tau, r_m >$.

6.2.13 First Row: $< \tau >$, $< \tau, \gamma >$. Second Row: $< \tau, \gamma, H >$ $< \tau, r_u >$. Third Row: $< \tau, r_m, H >$, $< \tau, H >$. Last Row: $\tau, r_m >$.

6.3 Wallpaper Groups

6.3.1 The first is rectangular, the second rhombal, and the third square.

6.3.3 The translation determined by f^2 will be in the same d rection as T, so we do not find two independent directions of tran lation.

6.3.5 The lattice for G will be invariant under rotations abou points of the lattice by a fixed angle. By the previous problem, thes rotations must be half-turns. By Theorem 6.14 the lattice must b Rectangular, Centered Rectangular, or Square.

6.3.7 Let C be the midpoint of the vector $v = \vec{AB}$, where v is o of the translation vectors for G. Let m_1 be a line perpendicular t $\overleftrightarrow{AB}$ at A. Then, $T_v = r_{m_1} \circ r_{m'_1}$ where m'_1 is a line perpendicular t $\overleftrightarrow{AB}$ at the midpoint of $\overline{AB}$. But since r_{m_1} is in G, then $r_{m_1} \circ T_v = r_{n}$ is in G. Likewise, we could find a line m'_2 perpendicular to th other translation vector $w = \vec{AC}$ at its midpoint, yielding anoth reflection $r_{m'_2}$. The formulas for these two reflections are $r_{m'_1}$ $r_{m_1} \circ T_v$ and $r_{m'_2} = r_{m_2} \circ T_w$.

6.3.9 In the exercise 6.3.8 we saw that the group of symmetri can be generated from reflections half-way along the translation ve tors. Thus, if we reflect the shaded region, we must get another pa of the pattern. Thus, three reflections of the shaded area will fill u the rectangle determined by v and w and the rest of the pattern wi be generated by translation.

6.3.11 If $A = lv + mw$ and $B = sv + tw$, then $0 \le s, t \le$ The length between A and B is the length of the vector $\vec{B - A}$ $(s-l)v+(t-m)w$. This length squared is the dot product of $\vec{B -}$ with itself, i.e., $(s-l)^2(v \bullet v)+2(s-l)(t-m)(v \bullet w)+(t-m)^2(w \bullet w$ If $v \bullet w > 0$, then this will be maximal when both $(s-l)$ and $(t-m$ are maximal. This occurs when $(s-l) = 1$ and $(t-m) = 1$, whic holds only if $s = 1 = t$ and $l = m = 0$. If $v \bullet w < 0$, we need $(s-l)$ t be as negative as possible, and $(t-m)$ to be as positive as possib (or vice-versa). In either case, we get values of 0 or 1 for s, t, l, an m.

6.3.13 A single straight line would have translational symmetri of arbitrarily small size.

6.5 Project 9 - Constructing Tessellations

Tiling is a fascinating subject. If you would like to know more abou the mathematics of tiling, a good supplementary source is *Tiling and Patterns*, by Grunbaum and Shephard.

A modern master of the art of tiling is M.C. Escher. A goo resource for his work is Doris Schattschnieder's book *M. C. Esche Visions of Symmetry.*

6.5.1 The symmetry group is $p4$.

Chapter 7

Non-Euclidean Geometry

The discovery of non-Euclidean geometry is one of the most impo
tant events in the history of mathematics. The book by Boyer an
Merzbach and the University of St. Andrews web site, both liste
in the bibliography of the text, are excellent references for a deep
look at this history.

Solutions to Exercises in Chapter 7

In section 7.2 we see for the first time the relevance of our earli
discussion of models in Chapter 1. The change of axioms in Chap
ter 7 (replacing Euclid's fifth postulate with the hyperbolic parall
postulate) requires a change of models. As you work through th
section, it is important to recall that, in an axiomatic system, it
not important what the terms actually mean; the only thing tha
matters is the relationships between the terms.

We introduce two different models at this point to help you rec
ognize the abstraction that lies behind the concrete expression o
points and lines in theses models.

7.2.2 Mini-Project: The Klein Model

It may be helpful to do the constructions (lines, etc) of the Klei
model on paper as you read through the material.

7.2.1 Use the properties of Euclidean segments.

7.2.3 The special case is where the lines intersect at a boundar
point of the Klein disk. Otherwise, use the line connecting the pol
of the two parallels to construct a common perpendicular.

7.3 Basic Results in Hyperbolic Geometry

In this section it is important to note the distinction between poin
at infinity and regular points. Omega triangles share some prop
erties of regular triangles, like congruence theorems and Pasch-lik
properties, but are not regular triangles—thus necessitating the th
orems found in this section.

7.3.1 Use the interpretation of limiting parallels in the Klei
model.

7.3.3 First, if m is a limiting parallel to l through a point
then $r_l(m)$ cannot intersect l, as if it did, then $r_l^2(m) = m$ woul
also intersect l. Now, drop a perpendicular from $r_l(P)$ to l at
and consider the angle made by Q, $r_l(P)$, and the omega point
$r_l(m)$. If there were another limiting parallel (n) to l through $r_l(P$
that lies within this angle, then by reflecting back by r_l we woul
get a limiting parallel $r_l(n)$ that lies within the angle made by
P and the omega point of l, which is impossible. Thus, $r_l(m)$ mus
be limiting parallel to l and reflection maps omega points to omeg
points, as r_l maps limiting parallels to l to other limiting paralle
Also, it must fix the omega point, as it maps limiting parallels
one side of the perpendicular dropped to l to limiting parallels
that same side.

7.3.5 Let P be the center of rotation and let l be a line throug
P with the given omega point Ω. (Such a line must exist as
must correspond to a limiting parallel line m, and there is always
limiting parallel to m through a given point P) Then, we can wri

$R = r_n \circ r_l$ for another line n passing through P. But, since r_l fixe Ω, and R does as well, then, r_n must fix Ω. But, if n and l are no coincident, then n is not limiting parallel to l and thus cannot hav the same omega points as l. By the previous exercise, r_n could no fix Ω. Thus, it must be the case that n and l are coincident and is the identity.

7.3.7 Let $PQ\Omega$ be an omega triangle and let R be a point interio to the triangle. Drop a perpendicular from Q to $\overleftrightarrow{P\Omega}$ at S. Then either R is interior to triangle QPS, or it is on $\overleftrightarrow{QS}$, or it is interio to $\angle QS\Omega$. If it is interior to ΔQPS it intersects $\overleftrightarrow{P\Omega}$ by Pasch axiom for triangles. If it is on $\overleftrightarrow{QS}$ it obviously intersects $\overleftrightarrow{P\Omega}$. If is interior to $\angle QS\Omega$, it intersects $\overleftrightarrow{P\Omega}$ by the definition of limitin parallels.

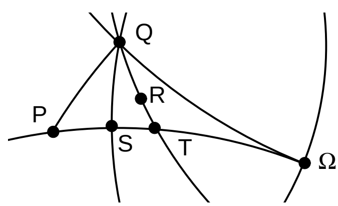

Figure 7.1:

7.3.9 Let l be the line passing through R. Then, either l passe within Omega triangle $PR\Omega$ or it passes within $QR\Omega$. In either cas we know by Theorem 7.5 that l must intersect the opposite side, i. it must intersect $\overline{P\Omega}$ or $\overline{Q\Omega}$.

7.3.11 Suppose we had another segment $\overline{P'Q'}$ with $\overline{P'Q'} \cong \overline{P}$ and let l' be a perpendicular to $\overline{P'Q'}$ at Q'. Let $\overleftrightarrow{P'R'}$ be a limitin parallel to l' at P'. Then, by Theorem 7.8, we know that $\angle QPR$ $\angle Q'P'R'$ and thus, the definition of this angle only depends on the length of $\overline{PQ}$.

7.3.13 Suppose $a(h) = a(h')$ with $h \neq h'$. We can assum that $h < h'$. But, then the previous exercise would imply tha $a(h) > a(h')$. Thus, if $a(h) = a(h')$ then $h = h'$.

7.4 Project 10 - The Saccheri Quadrilateral

As you do the computer construction of the Saccheri Quadrilatera you may experience a flip of orientation for your construction whe moving the quad about the screen. The construction depends c the orientation of the intersections of circles and these may switc as the quad is moved. A construction of the Saccheri quad that do not have this unfortunate behavior was searched for unsuccessfull by the author. A nice challenge problem would be to see if you ca come up with a better construction. If you can, the author woul love to hear about it!

7.4.1 Show that ΔADB and ΔBCA are congruent, and the show that ΔADC and ΔBDC are congruent.

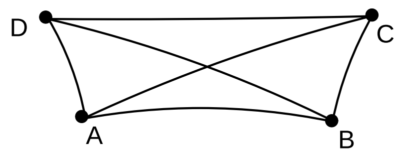

Figure 7.2:

7.5 Lambert Quadrilaterals and Triangles

7.5.1 Referring to figure 7.6, we know ΔACB and ΔACE are cor gruent by SAS. Thus, $\angle ACB \cong \angle ECA$. Since $\angle ACD \cong \angle FC$ and both are right angles, then $\angle BCD \cong \angle FCE$. Then, ΔBC and ΔFCE are congruent by SAS. We conclude that $\overline{BD} \cong \overline{F}$ and the angle at E is a right angle.

7.5.3 Create two Lambert quadrilaterals from the Saccheri quac lateral, and then use Theorem 7.13.

7.5.5 Since the angle at O is acute, then OAA' and OBB' ar triangles. Also, since $OA < OB$, then A is between O and B, an likewise A' is between O and B'. Thus, the perpendicular n at A t $\overleftrightarrow{AA'}$ will enter $\Delta OBB'$. By Pasch's axiom it must intersect $\overline{OB'}$ c

$\overline{BB'}$. It cannot intersect $\overline{OB'}$ as n and $\overleftrightarrow{OB'}$ must be parallel. Thu n intersects $\overline{BB'}$ at C. Then, $A'ACB'$ is a Lambert Quadrilatera and $B'C > A'A$. Since C is between B and B' we have $B'B > A'$

7.5.7 Let m be right limiting parallel to l at P and let P' b a point on m to the right of P (i.e. in the direction of the omeg point). Let Q and Q' be the points on l where the perpendicula from P and P' to l intersect l.

We claim that $m\angle QPP' < m\angle Q'P'R$ where R is a point on m t the right of P'. If these angles were equal we would have $\overline{PQ} \cong \overline{P'Q}$ by Exercise 7.3.11, and thus $QPP'Q'$ would be a Saccheri quadr lateral, which would imply that $\angle Q'P'R$ is a right angle, which impossible. If $m\angle QPP' > m\angle Q'P'R$, then $PQ < P'Q'$ by exercis 7.3.12, which would imply that we could find a point S on $\overline{P'Q'}$ wit $PQ = Q'S$, yielding Saccheri quadrilateral $PQQ'S$. Then, $\angle PSQ$ must be acute, which contradicts the Exterior angle theorem f $\Delta PSP'$.

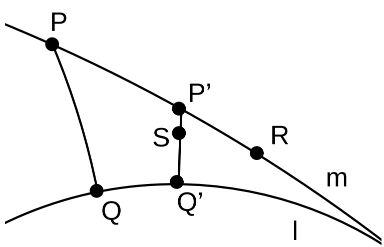

Figure 7.3:

Thus, $m\angle QPP' < m\angle Q'P'R$, and the result follows from exe cise 7.3.12.

7.5.9 If they had more than one common perpendicular, the we would have a rectangle.

7.5.11 Suppose Saccheri Quadrilaterals $ABCD$ and $EFGH$ hav $\overline{AB} \cong \overline{EF}$ and $\angle ADC \cong \angle EHG$. If $EH > AD$ then we can find on $\overline{EH}$ and J on $\overline{FG}$ such that $\overline{EI} \cong \overline{FJ} \cong \overline{AD}$. Then, by repeate application of SAS on sub-triangles of $ABCD$ and $EIJF$ we ca show that these two Saccheri Quadrilaterals are congruent. Bu this implies that the angles at H and I in quadrilateral $IHGJ$ ar supplementary, as are the angles at G and J, which means that w can construct a quadrilateral with angles sum of 360. This contra dicts Theorem 7.15, by considering triangles created by a diagona of $IHGJ$.

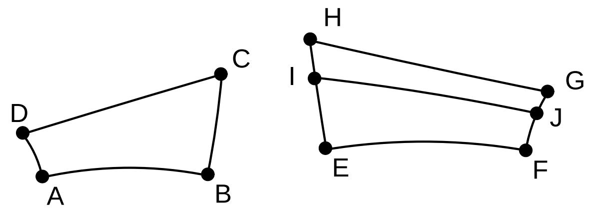

Figure 7.4:

7.5.13 No. To construct a scale model, we are really constructin a figure similar to the original. That is, a figure with correspondin angles congruent, and length measurements proportional by a no unit scale factor. But, Theorem 7.18 implies that any such sca model must have lengths preserved.

7.6 Area in Hyperbolic Geometry

In this section we can refer back to the mini-project we did on are in Chapter 2. That discussion depended on rectangles as the bas for a definition of area. In hyperbolic geometry, no rectangles exis so the next best shape to base area on is the triangle. This explain the nature of the theorems in this section.

7.6.1 Let J be the midpoint of $\overline{A''B}$ and suppose that $\overleftrightarrow{EF}$ cu $\overline{A''B}$ at some point $K \neq J$. Then, on $\overleftrightarrow{E''J}$ we can construct a secon Saccheri Quadrilateral by the method of dropping perpendicula from B and C to $\overleftrightarrow{E''J}$. Now, $\overline{BC}$ is the base of the original Sacche Quadrilateral $BCIH$ and the new Saccheri Quadrilateral. Thus, if

is the perpendicular bisector of $\overline{BC}$, then n meets $\overleftrightarrow{E''F}$ and $\overleftrightarrow{E''J}$ a
right angles. Since E'' is common to both curves, we get a triang
having two right angles, which is impossible.

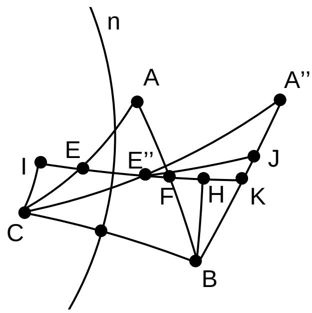

Figure 7.5:

7.6.3 This question can be argued both ways. If we could mak
incredibly precise measurements of a triangle, then we could po
sibly measure the angle sum to be less than 180. However, sinc
the universe is so vast, we would have to have an incredibly larg
triangle to measure, or incredibly good instruments. Also, we coul
never be sure of errors in the measurement overwhelming the actu
differential between the angle sum and 180.

7.7 Project 11 - Tiling the Hyperbolic Plane

A nice artisitic example of hyperbolic tilings can be found in M. C
Escher's Circle Limit figures. Consult Doris Schattschnieder's boo
M. C. Escher, Visions of Symmetry for more information abou
these tilings.

7.7.1 Reasoning as we did on Page 261 of the text, we see tha
if we have k regular n-gons meeting at a common vertex, then

$$180n < 360 + 2n\alpha$$

where $\alpha = \frac{360}{2k}$. Then,

$$\frac{360}{k} > 180 = \frac{360}{n}$$

and dividing by 360 and re-arranging gives

$$\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$$

Thus, since $\frac{1}{3} + \frac{1}{3} > \frac{1}{2}$ we have that a $(3, 3)$ tiling is possible.

7.7.3 In a $(6, 5)$ tiling we have regular hexagons meeting 5 a a vertex. The interior angles of the hexagons must be $\frac{360}{5} = 7$ Triangulating such a hexagon by triangles to the center, we see tha the central angle must be 60 degrees and the base angles of th isosceles triangles must be 36 degrees (half the interior angle).

Thus, to build the tiling we start with a triangle of angles 6 36, and 36 and continue the construction just as we did in the lab

Chapter 8

Non-Euclidean Transformations

In this chapter we extend our notion of isometry from Euclidea
geometry to hyperbolic geometry. The discussion on pages 316-31
is intended to make the subsequent focus on Möbius transformatio
a natural condition for carrying out this extension.

Section 8.1 might seem to be a side-track, but it is necessa
groundwork material needed to put the subsequent development
isometries on a firm footing.

Solutions to Exercises in Chapter 8

8.2 Isometries in the Poincaré Model

In this section we see what isometries look like in the Poinca
Model. We use the principles of Klein's Erlanger Programm her
That is, we are aboe to prove general results about figures by tran
forming the figures to "nice" locations and proving the result ther

8.2.1 Let G be the set of rigid motions. Let $f(z) = e^{i\phi_1}z + b_1$ an
$g(z) = e^{i\phi_2}z + b_2$. Then, $g \circ f(z) = e^{i\phi_2}(e^{i\phi_1}z + b_1) + b_2 = e^{i\phi_1+\phi_2}z$
$(e^{i\phi_2}b_1 + b_2)$ and thus $g \circ f(z)$ is in G. Since $f^{-1}(z) = e^{-i\phi_1}z - e^{-i\phi_1}$

then f^{-1} is in G. If $\phi = 0$ and $b = 0$ we get the identity in G. Lastl associativity is automatic, as function composition is associative.

8.2.3 By Theorem 8.3 we can find a transformation such tha the points 1,−1, and i go to 1, ∞, and 0. This transformation $f(z) = \frac{z-i}{z+1}\frac{2}{1-i}$. Also, $g(z) = \frac{z}{z+1}\frac{2}{1}$ takes 1, −1, and 0 to 1, ∞, an 0. Then, $g^{-1} \circ f$ is the desired transformation by Theorem 8.7.

8.2.5 Choose $a = 1 = d$ and $b = c = 0$ in $f(z) = \frac{ax+b}{cz+d}$.

8.2.7 Because function composition is associative.

8.2.9 Let $P = z_0$ and $Q = z_1$. Let $f(z) = \frac{z-z_0}{\overline{z_0}-1}$. Then, $f(z_0)$ 0 and f is a hyperbolic isometry. Next, let $g(z) = \frac{z-z_1}{\overline{z_1}-1}$. Ther $g(z_1) = 0$ and g is a hyperbolic isometry. The composition $g^{-1} \circ$ maps P to Q.

8.2.11 Let l be a hyperbolic line from P to Q. We can find hyperbolic transformation that maps P to the origin (see the expla nation for exercise 8.2.9). If the transformed line does not lie alon the axis, we can transform it to the axis by a rotation. The cros ratio is invariant under both of these transformations. Clearly, th cross-ratio defined for points on the x-axis is real. Also, the cros ratio will look like $\frac{1}{-1}\frac{b+1}{b-1}$ which is always positive.

8.2.13 Since d_H is invariant under hyperbolic isometries we hav $d_H(z_0, z_1) = d_H(g(z_0), g(z_1))$. Since, $g(z_1) = 0$ we have by Theore 6.12

$$
\begin{aligned}
d_H(z_0, z_1) &= ln(\frac{1 + |g(z_0)|}{1 - |g(z_0)|}) \\
&= ln(\frac{1 + \frac{|z_0 - z_1|}{|1 - z_0\overline{z_1}|}}{1 - \frac{|z_0 - z_1|}{|1 - z_0\overline{z_1}|}})
\end{aligned}
$$

Finding a common denominator in the last equation yields th result.

8.2.15 As in the proof of Theorem 8.11, we know that

$$
\begin{aligned}
d_P(z_0, z_1) &= |\ln(\frac{z_0 - w_1}{z_0 - w_0}\frac{z_1 - w_0}{z_1 - w_1})| \\
&= |\ln((z_0, z_1, w_1, w_0))
\end{aligned}
$$

for z_0 and z_1 in the disk, where w_0 and w_1 are the points of interse tion of the circle through z_0 and z_1 (call this circle c) with the un circle. Let z_0^* and z_1^* be the inverse points of z_0 and z_1 with respe to c. Then, by the proof of Lemma 8.8 we have

$$\begin{aligned} d_P(z_0^*, z_1^*) &= |\ln((z_0^*, z_1^*, w_1, w_0))| \\ &= |\ln(\overline{(z_0, z_1^*, w_1, w_0)})| \end{aligned}$$

But, $\overline{(z_0, z_1^*, w_1, w_0)} = (z_0, z_1^*, w_1, w_0)$, as z_0, z_1, w_0, and w_1 lie c the same circle. Thus,

$$\begin{aligned} d_P(z_0^*, z_1^*) &= |\ln((z_0, z_1^*, w_1, w_0))| \\ &= |-\ln((z_1^*, z_0, w_1, w_0))| \\ &= |\ln((z_1^*, z_0, w_1, w_0))| \\ &= |\ln(\overline{(z_1, z_0, w_1, w_0)})| \end{aligned}$$

Again, $\overline{(z_1, z_0, w_1, w_0)} = (z_1, z_0, w_1, w_0)$ and so,

$$\begin{aligned} d_P(z_0^*, z_1^*) &= |\ln((z_1, z_0, w_1, w_0))| \\ &= |-\ln((z_0, z_1, w_1, w_0))| \\ &= d_P(z_0, z_1) \end{aligned}$$

To show that inversion is a reflection across c, we just note tha inversion preserves the circle of inversion, and thus fixes the Poinca line defined by c.

8.3 Isometries in the Klein Model

In section 8.2 we see isometries treated in a very functional way—w have formulas for isometries in the Poincaré disk defined by comple rational functions. This section serves as a nice contrast in tha isometries will be defined in a very geometric way through the us of poles. Also, isometries are defined by starting with reflections, i the same way isometries were developed in Chapter 5.

8.3.1 Let t be the Klein line through P and P', and construc the pole of t. Let Ω be the omega point where the Euclidean lin through the pole of t and P meets the boundary circle (on the othe side of t from the pole of t). Let Ω' be the omega point where th Euclidean line through the pole of t and P' meets the boundar circle (on the same side of t from the pole of t). (Refer to Fig. 8.1 Then, the point Q where $\overline{\Omega\Omega'}$ intersects t is the midpoint of $\overline{PP}$ This can be seen by using the angle-angle congruence theorem fo Omega triangles. The line l through Q and the pole of t will be th perpendicular bisector of $\overline{PP'}$.

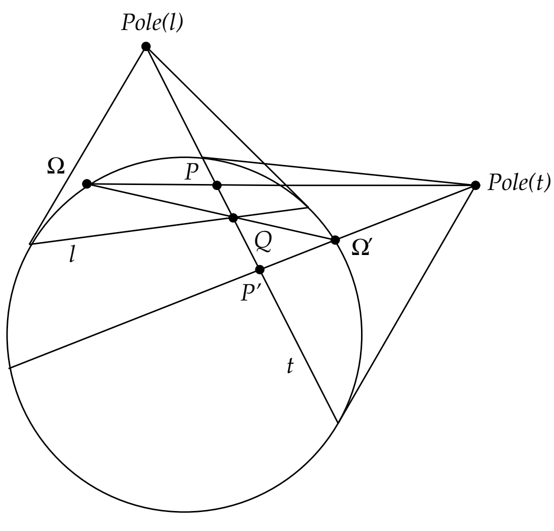

Figure 8.1:

8.3.3 One possible construction is illustrated in Fig. 8.2. Le $\overline{AB}$ be a diameter of the Klein disk and let A be a point not at th center. Let B be the reflection of A across a diameter perpendicula to $\overline{AB}$ and construct two Klein lines (l and m) at A and B that ar perpendicular to $\overline{AB}$. Construct the poles C and D to these line and let $\overrightarrow{CE}$ be a ray from C intersecting l at E. This ray will creat

Klein line n. Then, there will be a common perpendicular ($\overleftrightarrow{GH}$) t
m and n, using the result from Ex 8.3.2. Also, this line must be o
the same side of $\overline{AB}$ as E is. Then, $AEGHB$ is a pentagon wit
five right angles.

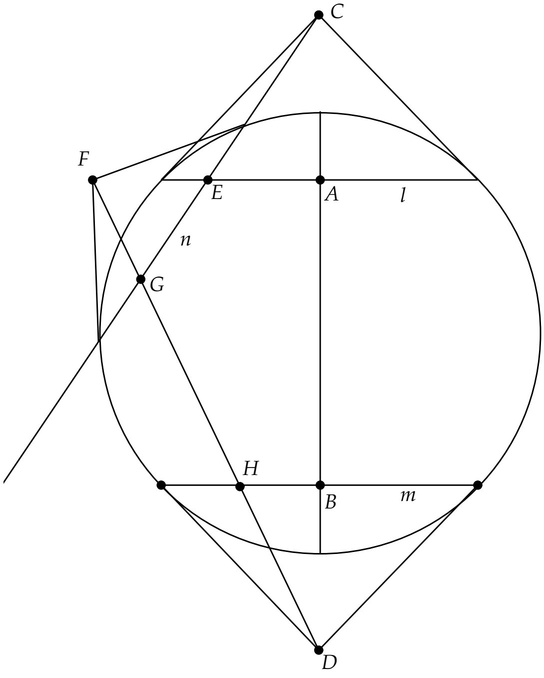

Figure 8.2:

8.3.5 We know from the construction of a Klein reflection that
will map a point P to a point P' that lies on a line perpendicular t
l. Thus, if P is already on a perpendicular t to l, then its reflectio
is again on t. Likewise, $r_m(r_l(P))$ is again on t.

8.4 Mini-Project: The Upper Half-Plane Model

In this project we see yet a third model for hyperbolic geometr
A significant new development in this section is the idea of mod

isomorphism. It would be a good idea to review this idea before yo start this project.

8.4.1 There are two cases. If $c = 0$, then $f(z) = a'z + b$ where $a' = \frac{a}{d}$ and $b' = \frac{b}{d}$. Since $f(0) = b'$, then b' is real. Sinc $f(1) = a' + b''$ is real then a' is real. Clearly, we can assume $d =$ in the fraction defining f. Thus, a,b,c, and d are real.

If $c \neq 0$, then we can again assume $c = 1$ by dividing top an bottom of the fraction by c. Since $f(0) = \frac{b}{d} = r_1$ is real, the $b = r_1 d$. Also, since $f(\infty) = \frac{a}{c} = r_2$ is real, then $a = r_2 c$. Thu $f(z) = \frac{r_2 cz + r_1 d}{cz+d}$. Now, for some real r_3 we have $f(r_3) = \frac{r_2 c r_3 + r_1 d}{c r_3 + d}$ ∞. Thus, $cr_3 + d = 0$ or $d = -r_3 c$. Then,

$$
\begin{aligned}
f(z) &= \frac{r_2 cz + r_1(-r_3 c)}{cz - r_3 c} \\
&= \frac{r_2 cz - r_1 r_3 c}{cz - r_3 c} \\
&= \frac{r_2 z - r_1 r_3}{z - r_3}
\end{aligned}
$$

Comparing this fraction with the original we see that a,b,c, an d are real.

8.4.3 If we consider the x-axis as the equivalent of the Poincar circle, then "lines" should be clines that meet this boundary at righ angles. That is, lines should be either Euclidean lines that are pe pendicular to the x-axis, or arcs of circles perpendicular to the axi That is, semi-circles with centers along the axis.

8.4.5 You can argue that any configuration of a "line" and point off the line can be transformed by a suitable upper half-plan transformation to the scene illustrated in Figure 8.4. Clearly, the are an infinite number of semi-circles through z_0 that do not interse the y-axis.

8.6 Hyperbolic Calculation

In this section we do some basic calculus of hyperbolic geometr Klein's transformational view really shines here. We see how to d velop some exceptionally nice formulas for arclength, the angle parallelism, and area using proofs based on simple configuration Also, the hyperbolic Pythagorean Theorem is a nice result in th section. The fact that hyperbolic geometry is "locally Euclidean can be demonstrated nicely with the hyperbolic Pythagorean The orem. If we compute the Taylor expansion for cosh we see tha $\cosh(c) = \cosh(a)\cosh(b)$ has as its second-order approximation th Euclidean Pythagorean Theorem.

8.6.1 Use the definition of cosh and sinh.

8.6.3 This is a simple matter of checking the algebra.

8.6.5 Since the map S preserves the two distance functions i the models, then the lengths of the curves must be the same.

Next,

$$\begin{aligned} |z'| &= \left|\frac{(w+i)-(w-i)}{(w+i)^2}\right| |w'| \\ &= \frac{2}{|w+i|^2}|w'| \end{aligned}$$

Thus, using the change of variable formula for integration, w get

$$\begin{aligned} \int_a^b \frac{2|z'(t)|}{1-|z|^2}dt \\ &= \int_a^b \frac{\frac{4|w'(t)|}{|w+i|^2}}{1-\frac{|w-i|^2}{|w+i|^2}}dt \\ &= \int_a^b \frac{4|w'(t)|}{|w+i|^2-|w-i|^2}dt \\ &= \int_a^b \frac{4|w'(t)|}{(w+i)(\overline{w}-i)-(w-i)(\overline{w}+i)}dt \\ &= \int_a^b \frac{4|w'(t)|}{2(-iw+i\overline{w})}dt \\ &= \int_a^b \frac{|w'(t)|}{v(t)}dt \end{aligned}$$

8.7 Project 12 - Infinite Real Estate?

You will probably not believe the results of this project, which mak it such a great lab!

8.7.1 We note that

$$\begin{aligned} S(w) &= i\,\frac{w-i}{w+i} \\ &= i\,\frac{(u+i(v-1))}{(u+i(v+1))} \\ &= i\,\frac{(u+i(v-1))(u-i(v+1))}{u^2+(v+1)^2} \\ &= i\,\frac{(u^2+(v^2-1))+i(-2u)}{u^2+(v+1)^2} \end{aligned}$$

Thus,

$$\begin{aligned} x &= \frac{2u}{u^2+(v+1)^2} \\ y &= \frac{u^2+(v^2-1)}{u^2+(v+1)^2} \end{aligned}$$

8.7.3 The angle θ will be defined by the tangent $\overleftrightarrow{PB}$ to the circ at P. If θ is 90 degrees, then this tangent is perpendicular to th y-axis and it is obvious that the angle in the Ω triangle at P is right angle.

Otherwise, we can assume the tangent intersects the x-axis B. It follows that ΔOPB is a right triangle with right angle at Drop a perpendicular from P to the x-axis, intersecting at A. The $\angle APB$ has measure θ. It immediately follows that the interior ang of the doubly limiting triangle at P has measure θ.

Lab Conclusion For the conclusion of the lab, note that a tr angular area in hyperbolic geometry has area bounded by π by Th orem 8.28. A 4-sided figure can be split into two triangular figure and so its area must be bounded by 2π. A five-sided figure wou have area bounded by 3π, etc.

Chapter 9

Fractal Geometry

Much of the material in this chapter is at an advanced level, espe-cially the sections on contraction mappings and fractal dimension–Sections 9.5 and 9.6. But this abstraction can be made quite con-crete by the computer explorations developed in the chapter. I fact, the computer projects are the *only* way to really understan these geometric objects on an intuitive level.

Solutions to Exercises in Chapter 9

9.3 Similarity Dimension

The notion of dimension of a fractal is very hard to make precis In this section we present one simple way to define dimension, bu there are also other ways to define dimension as well, each useful fo a particular purpose and all agreeing with integer dimension, bu not necessarily with each other.

9.3.1 Theorem 2.27 guarantees that the sides of the new triangl are parallel to the original sides. Then, we can use SAS congruenc to achieve the result.

9.3.3 At each successive stage of the construction, 8 new square are created, each of area $\frac{1}{9}$ the area of the squares at the previou stage. Thus, the pattern for the total area of each successive stag of the construction is

$$
\begin{aligned}
l &= 1 - \frac{1}{9} - \frac{8}{81} - \frac{64}{9^3} - \dots \\
&= 1 - \frac{1}{9} \sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k \\
&= 1 - \frac{1}{9} \frac{1}{1 - \frac{8}{9}} \\
&= 1 - 1 \\
&= 0
\end{aligned}
$$

Thus, the area of the final figure is 0.

9.3.5 The similarity dimension would be $\frac{\log(4)}{\log(3)}$.

9.3.7 Split a cube into 27 sub-cubes, as in the Menger spong construction, and then remove all cubes except the eight corn cubes and the central cube. Do this recursively. The resulting fract will have similarity dimension $\frac{\log(9)}{\log(3)}$, which is exactly 2.

9.4 Project 13 - An Endlessly Beautiful Snowflake

If you want a challenge, you could think of other templates base on a simple segment, generalizing the Koch template and the Ha template from exercise 9.4.4.

9.4.1 At stage 0 the Koch curve has length 1. At stage 1 it ha length $\frac{4}{3}$. At stage 2 it has length $\frac{16}{9} = \frac{4^2}{3^2}$, since each segment replaced by the template, which is $\frac{4}{3}$ as long as the original segmen Thus, at stage n the length will be $\frac{4^n}{3^n}$, and so the length will go t infinity.

9.4.3 The similarity dimension will be that of the template r placement fractal. The similarity ratio is $\frac{1}{3}$ and it takes 4 sub-objec to create the template. Thus, the similarity dimension is $\frac{\log(4)}{\log(3)}$.

9.6 Fractal Dimension

Sections 9.5 and 9.6 are quite "thick" mathematically. To get som sense of the Hausdorff metric, you can compute it for some simp pairs of compact sets. For example, two triangles in different po sitions. Ample practice with examples will help you get a feel fo the mini-max approach to the metric and this will also help you b successful with the homework exercises.

9.6.1 A function f is continuous if for each $\epsilon > 0$ we can fin $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ when $0 < |x - y| < \delta$. Let S be contraction mapping with contraction factor $0 \leq c < 1$. Then, give ϵ, let $\delta = \epsilon$ (if $c = 0$) and $\delta = \frac{\epsilon}{c}$ (if $c > 0$).

If $c = 0$ we have $0 = |S(x) - S(y)| \leq |x - y| < \delta = \epsilon$.

If $c > 0$, we have $|S(x) - S(y)| \leq c|x - y| < c\frac{\epsilon}{c} = \epsilon$.

9.6.3 Property (2): Since $d_{\mathcal{H}}(A, A) = d(A, A)$, and since $d(A, A$ $max\{d(x, A)|x \in A\}$, then we need to show $d(x, A) = 0$. Bu $d(x, A) = min\{d(x, y)|y \in A\}$, and this minimum clearly occu when $x = y$; that is, when the distance is 0.

Property (3): If $A \neq B$ then we can always find a point x i A that is not in B. Then, $d(x, B) = min\{d(x, y)|y \in B\}$ must b greater than 0. This implies that $d(A, B) = max\{d(x, B)|x \in A\}$ also greater than 0.

9.6.5 We know that

$$\begin{aligned}
d(A, C \cup D) &= max\{d(x, C \cup D)|x \in A\} \\
&= max\{min\{d(x, y)|x \in A \; and \; y \in C \; or \; D\}\} \\
&= max\{min\{min\{d(x, y)|x \in A \, y \in C\}, min\{d(x, y \\
&= max\{min\{d(x, C), d(x, D)\}|x \in A\}
\end{aligned}$$

The last expression is clearly less than or equal to $max\{d(x, C)|x$ $A\} = d(A, C)$ and also less than or equal to $max\{d(x, D)|x \in A\}$ $d(A, D)$.

9.6.7 There are three contraction mappings which are used t construct Sierpinski's triangle. Each of them has contraction sca

factor of $\frac{1}{2}$. Thus, we want $(\frac{1}{2})^D + (\frac{1}{2})^D + (\frac{1}{2})^D = 1$, or $3(\frac{1}{2})^D =$
Solving for D we get $D = \frac{log(3)}{log(2)}$.

9.7 Project 14 - IFS Ferns

Do not worry too about getting exactly the same numbers for th scaling factor and the rotations that define the fern. The importan idea is that you get the right *types* of transformations (in the corre order of evaluation) needed to build the fern image. For exercis 9.7.5, it may be hopeful to copy out one piece of the image and the rotate and move it so it covers the other pieces, thus generating th transformations needed.

9.7.1 The rotation matrix R is given by

$$\begin{bmatrix} \cos(\frac{5\pi}{180}) & \sin(\frac{5\pi}{180}) \\ -\sin(\frac{5\pi}{180}) & \cos(\frac{5\pi}{180}) \end{bmatrix} \approx \begin{bmatrix} 0.996 & 0.087 \\ -0.087 & 0.996 \end{bmatrix}$$

The scaling matrix S is given by

$$\begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}$$

If we let T be the translation in the vertical direction by h, the $T_1 = T \circ S \circ R$, which after rounding to the nearest tenth, match the claimed affine transformation in the text.

9.7.3 The rotation matrix R is given by

$$\begin{bmatrix} \cos(\frac{-60\pi}{180}) & -\sin(\frac{-60\pi}{180}) \\ \sin(\frac{-60\pi}{180}) & \cos(\frac{-60\pi}{180}) \end{bmatrix} \approx \begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix}$$

The scaling matrix S is given by

$$\begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}$$

The reflection matrix r is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we let T be the translation in the vertical direction by $\frac{h}{2}$, the $T_3 = T \circ S \circ R \circ r$, which after rounding to the nearest hundredth matches the claimed affine transformation in the text.

9.7.5 For the lower left portion of the shape, we need to sca the whole figure down by a little less than 0.5, say by 0.48. Als we need to rotate the figure by 90 degrees and then translate back by 0.5 in the x-direction to put it in place. Let T_1 be th net transformation accomplishing this. Then $T_1(x, y) = (-0.48y$ $0.5, 0.48x)$. Let T_2 be the transformation for the upper left portion Then $T_2(x, y) = (0.5x, 0.5y + 0.5)$. Let T_3 be the transformation fo the upper right portion. Then $T_3(x, y) = (0.48y + 0.5, -0.48x + 1.0$ Finally. let T_4 be the transformation for the small inner part. The $T_4(x, y) = (0.3x + 0.3, 0.3y + 0.3)$ would work.

9.9 Grammars and Productions

This section will be very different from anything you have done be fore, except for those who have had some computer science course The connection between re-writing and axiomatic systems is a dee one. One could view a theorem as essentially a re-writing of variou symbols and terms used to initialize a set of axioms. Also, turt geometry is a very concrete way to view re-writing and so we hav a nice concrete realization of an abstract idea.

9.9.1 Repeated use of production rule 1 will result in an ex pression of the form $a^n S b^n$. Then, using production rule 2, we ge $a^n b^n$.

9.9.3 The level 1 rewrite is $+RF - LFL - FR+$. This is show in Fig. 9.1. The level 2 rewrite is $+ - LF + RFR + FL - F - +RF$ $LFL - FR + F + RF - LFL - FR + -F - LF + RFR + FL - -$ This is shown in Fig. 9.2. For the last part of the exercise, note tha all interior "lattice" points (defined by the length of one segmen are actually visited by the curve. Thus, as the level increases (an we scale the curve back to some standard size) the interior poin will cover space, just as the example in section 9.9 did.

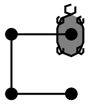

Figure 9.1:

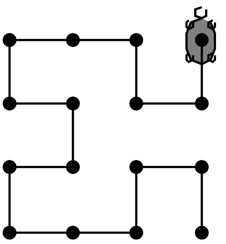

Figure 9.2:

9.10 Project 15 - Words Into Plants

Grammars as representations of growth is an idea that can be tied i nicely with the notion of genetics from biology. A grammar is like blueprint governing the evolution of the form of an object such as bush, in much the same way that DNA in its expression as protei governs the biological functioning of an organism.

9.10.1 The start symbol was rewritten twice.

9.10.3 Here's one simple example, plus the image generated fro rewriting to a level of 3 (Fig. 9.3).

Productions: $X- > F[+X][++X][-X][--X]X$ (Use a sma turn angle)

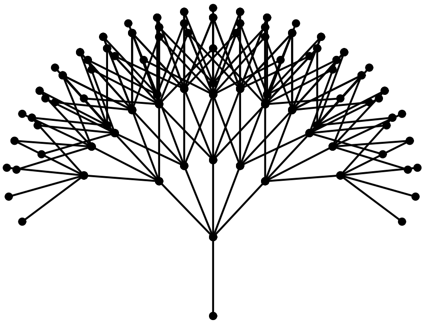

Figure 9.3:

Appendix A

Sample Lab Report

Pollie Gonn
MCS 303 Project 0
September 12, 2003
The Amazing Pythagorean Theorem

Introduction

The Pythagorean Theorem is perhaps the most famous theore in geometry, if not in all of mathematics. In this lab, we look a one method of proving the Pythagorean Theorem by constructin a special square. Part I of this report describes the constructio used in the proof and Part II gives a detailed explanation of wh this construction works, that is why the construction generates proof of the Pythagorean theorem. Finally, we conclude with som comments on the many proofs of the Pythagorean Theorem.

Part I:

To start out our investigation of the Pythagorean Theorem, w assume that we have a right triangle with legs b and a and hy potenuse c. Our first task construction is that of a segment sub divided into two parts of lengths a and b. Since a and b are a bitrary, we just create a segment, attach a point, hide the origin segment, and draw two new segments as shown.

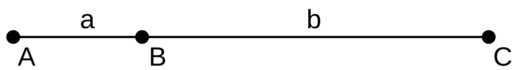

Figure A.1:

Then, we construct a square on side a and a square on side
The purpose of doing this is to create two regions whose total are
is $a^2 + b^2$. Clever huh? Constructing the squares involved severa
rotations, but was otherwise straightforward.

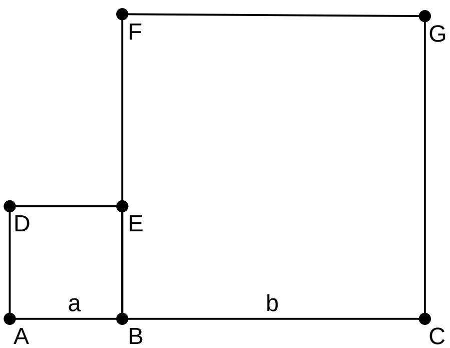

Figure A.2:

The next construction was a bit tricky. We define a translatio
from B to A and translate point C to get point H. Then, we connec
H to D and H to G, resulting in two right triangles. In part II, w
will prove that both of these right triangles are congruent to th
original right triangle.

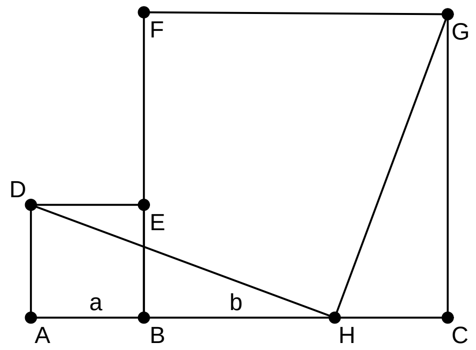

Figure A.3:

Next, we hide segment BC and create segments BH and H(
This is so that we have well-defined triangle sides for the next ste
- rotating right triangle ADH 90 degrees about its top vertex, an
right triangle HGC -90 degrees about its top vertex.

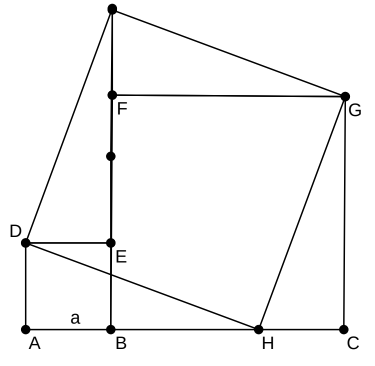

Figure A.4:

Part II:

We will now prove that this construction yields a square (on DH
of side length c, and thus, since the area of this square is clear
equal to the sum of the areas of the original two squares, we hav

$a^2 + b^2 = c^2$, and our proof would be complete. By SAS, triang HCB must be congruent to the original right triangle, and thus i hypotenuse must be c. Also, by SAS, triangle DAH is also congruer to the original triangle, and so its hypotenuse is also c. Then, angl AHD and CHG(= ADH) must sum to 90 degrees, and the ang DHG is a right angle. Thus, we have shown that the constructio yields a square on DH of side length c, and our proof is complete.

Conclusion:

This was a very elegant proof of the Pythagorean Theorem. I researching the topic of proofs of the Pythagorean Theorem, we di covered that over 300 proofs of this theorem have been discovere Elisha Scott Loomis, a mathematics teacher from Ohio, compile many of these proofs into a book titled *The Pythagorean Propos tion*, published in 1928. This tidbit of historical lore was gleane from the Ask Dr. Math website (http://mathforum.org/library/drmath/view/62539.html). It seen that people cannot get enough of proofs of the Pythagorean Theo rem.

Appendix B

Sample Lab Grading Sheets

Sample Grade Sheet for Project 1 - The Ratio Made o Gold

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriat headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 50 points - Discussion of Project Work and Solutions to Exe cises
  - 5 Discussion of the Construction of the Golden Ratio
  - 5 Discussion of the Construction of the Golden Rectang
  - 10 Solution to Exercise 1.3.1
  - 10 Solution to Exercise 1.3.2
  - 10 Solution to Exercise 1.3.3

  - 10 Solution to Exercise 1.3.4

- Total Points for Project (out of 60 possible)

Sample Grade Sheet for Project 2 - A Concrete Ax iomatic System

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropria headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 50 points - Discussion of Project Work and Solutions to Exe cises
  - 10 Discussion of Euclid's Five Postulates
  - 10 Construction of Rectangles
  - 10 Sum of Angles in a Triangle
  - 10 Euclid's Equilateral Triangle Construction
  - 10 Perpendicular to a Line through a Point Not on th Line
- Total Points for Project (out of 60 possible)

Sample Grade Sheet for Project 3 - Special Points of Triangle

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriat headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 60 points - Discussion of Project Work and Solutions to Exe cises
  - 10 Discussion of Work Done in Lab
  - 10 Exercise 2.3.1
  - 10 Exercise 2.3.2
  - 10 Exercise 2.3.3
  - 10 Exercise 2.3.4
  - 10 Exercise 2.3.5
- Total Points for Project (out of 70 possible)

Sample Grade Sheet for Project 4 - Circle Inversion an Orthogonality

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriat headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 50 points - Discussion of Project Work and Solutions to Exe cises
  - 10 Discussion of Work Done in Lab
  - 10 Exercise 2.7.1
  - 10 Exercise 2.7.2
  - 10 Exercise 2.7.3
  - 10 Exercise 2.7.4
- Total Points for Project (out of 60 possible)

Sample Grade Sheet for Project 7 - Quilts and Transformations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriate headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 50 points - Discussion of Project Work and Solutions to Exercises
  - 10 Discussion of Initial Work Done on Quilt 1
  - 10 Exercise 4.5.1
  - 10 Exercise 4.5.2
  - 10 Exercise 4.5.3
  - 10 Exercise 4.5.4
- Total Points for Project (out of 60 possible)

Sample Grade Sheet for Project 8 - Constructing Com positions

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriat headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 50 points - Discussion of Project Work and Solutions to Exe cises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 4.8.1
  - 10 Exercise 4.8.2
  - 10 Exercise 4.8.3
  - 10 Exercise 4.8.4
- Total Points for Project (out of 60 possible)

Sample Grade Sheet for Project 9 - Constructing Tesse-lations

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropria headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 30 points - Discussion of Project Work and Solutions to Exe-cises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 5.5.1
  - 10 Exercise 5.5.2
- Total Points for Project (out of 40 possible)

Sample Grade Sheet for Project 10 - The Saccheri Quad lateral

- 10 points - Organization And Writing Mechanics
  - 5 Structure of report is clear, with logical and appropriat headings and captions, including an introduction and conclusion.
  - 5 Spelling and Grammar
- 60 points - Discussion of Project Work and Solutions to Exe cises
  - 10 Discussion of Initial Work Done
  - 10 Exercise 6.5.1
  - 8 Exercise 6.5.2 part i
  - 8 Exercise 6.5.2 part ii
  - 8 Exercise 6.5.2 part iii
  - 8 Exercise 6.5.2 part iv
  - 8 Exercise 6.5.2 part v
- Total Points for Project (out of 70 possible)