CHAPTER 16

Complex Analytic Functions

> The shortest route between two truths in the real domain passes through the complex domain.
>
> – Jacques Hadamard (1865-1963)

16.1 THE COMPLEX PLANE

Complex numbers are numbers of the form

$$\alpha = a + ib$$

where a and b are real. The set of all possible numbers of this form will be called the set of *complex numbers* and the plane containing these numbers will be called the *complex plane*.

A complex number has an interesting dual nature. It can be thought of geometrically as a vector (i.e., an ordered pair of numbers), or it can be thought of algebraically as a single (complex) number having real components.

Definition 16.1. *If $z = x+iy$ is a complex number, then x is called the* real part *of z, denoted $Re(z)$, and y is called the* imaginary part, *denoted $Im(z)$.*

Given a complex number $z = x + iy$, or two complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$, we define basic algebraic operations as follows:

Addition-Subtraction $\quad z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

Multiplication by Real Scalar $kz = kx + iky$, for k a real number

Multiplication $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Complex Conjugate $\overline{z} = x - iy$

Modulus $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$

Note that complex addition (subtraction) is defined so that this operation satisfies the definition of vector addition (subtraction).

The modulus of z is the same as the norm (length) of the vector that z represents. The conjugate of z yields a number that is the reflection of z (considered as a vector) across the x-axis.

16.1.1 Polar Form

The x and y coordinates of a complex number z can be written as

$$x = r\ \cos(\theta),\ y = r\ \sin(\theta)$$

where r is the length of $v = (x, y)$ and θ is the angle that v makes with the x-axis. Since $r = |z|$, then

$$z = x + iy = |z|(\cos(\theta) + i\ \sin(\theta))$$

The term $(\cos(\theta) + i\ \sin(\theta))$ can be written in a simpler form using the following definition.

Definition 16.2. *The complex exponential function e^z is defined as*

$$e^z = e^{x+iy} = e^x(\cos(y) + i\ \sin(y))$$

From this definition we can derive *Euler's Formula*:

$$e^{i\theta} = \cos(\theta) + i\ \sin(\theta)$$

Thus, the polar form for a complex number z can be written as

$$z = |z|e^{i\theta}$$

All of the usual power properties of the real exponential hold for e^z, for example, $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Thus,

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$

We also note that if $z = |z|e^{i\theta}$, then

$$\overline{z} = |z|e^{-i\theta}$$

The angle coordinate for z will be identified as follows:

Definition 16.3. *Given $z = |z|e^{i\theta}$, the* argument *or* arg *of z is a value between 0 and 2π defined by*

$$arg(z) = \theta \,(\text{mod } 2\pi)$$

We use here the modular arithmetic definition that $a \bmod n$ represents the remainder (in $[0, n)$) left when a is divided by n. For example, $24 \bmod 10$ is 4.

From the definition of arg, and using the properties of the complex exponential, we see that if z and w are complex numbers, then

$$arg(zw) = (arg(z) + arg(w))(\text{mod } 2\pi)$$

The proof is left as an exercise. Also, if $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$, then $wz = |w||z|e^{i(\phi+\theta)}$.

16.1.2 Complex Functions

A complex function f in a region R of the plane is a rule that assigns to every $z \in R$ a complex number w. The relationship between z and w is designated by $w = f(z)$. In the last section, $f(z) = e^z$ defined a complex function on the entire complex plane.

Every complex function is comprised of two real-valued functions. By taking the real and imaginary parts of $w = f(z)$, we get that

$$f(x + iy) = u(x, y) + iv(x, y)$$

For example, if $f(z) = z^2$, then $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

One of the simplest classes of complex functions is the set of polynomials with complex coefficients. One of the most significant results in the area of complex numbers is that every complex polynomial has at least one root, and therefore has a complete set of roots (see [17] for a proof).

Theorem 16.1. *(Fundamental Theorem of Algebra) Let $p(z)$ be a non-constant polynomial. Then, there is a complex number a with $p(a) = 0$.*

The Point at Infinity and The Extended Complex Plane

The set of points for which the function $w = \frac{1}{z}$ is defined will include all complex numbers, except $z = 0$. As z approaches 0, the modulus of w will increase without bound.

Also, for all $w \neq 0$ there is a point z for which $w = \frac{1}{z}$. Thus, $f(z) = \frac{1}{z}$ defines a one-to-one function from the complex plane (minus $z = 0$) to the complex plane (minus $w = 0$).

We call a function f that maps a set S to a set S' *one-to-one* (1-1) if it has the property that whenever $f(s) = f(t)$, then $s = t$, for s and t in S. We will call f *onto* if for all elements s' in S', there is an s in S such that $f(s) = s'$.

In order to make f a function defined on all points of the complex plane, we extend the complex plane by adding a new element, the *point at infinity*, denoted by ∞. To be more precise, we define the point at infinity as follows:

Definition 16.4. *The* point at infinity *is the limit point of every sequence $\{z_n\}$ of complex numbers that is increasing without bound. A sequence is increasing without bound if for all $L > 0$ we can find N such that $|z_n| > L$ for all $n > N$.*

What properties does the point at infinity have? If $\{z_n\}$ increases without bound, then $\{\frac{1}{z_n}\}$ must converge to zero. So, if $\infty = \lim_{n\to\infty} z_n$, then $0 = \lim_{n\to\infty} \frac{1}{z_n}$.

Thus, it makes sense to define $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. Then, $f(z) = \frac{1}{z}$ will be a one-to-one map of the *extended* complex plane (the complex plane plus the point at infinity) onto itself.

Whereas we can conceptualize the set of complex numbers as the Euclidean (x, y) plane, the extended complex plane, with an ideal point at infinity attached, is harder to conceptualize. It turns out that the extended complex plane can be identified with a sphere through a process called *stereographic projection*.

Stereographic Projection

In a three-dimensional Euclidean space with coordinates (X, Y, Z), let S be the unit sphere, as shown in Figure 16.1.

Let N be the north pole of the sphere, the point at $(0, 0, 1)$. The sphere is cut into two equal hemispheres by the X-Y plane, which we will identify with the complex plane. Given a point $P = z = (x + iy)$ in the complex plane, we map P onto the sphere by joining N to P by a line and finding the intersection point P' of this line with the sphere.

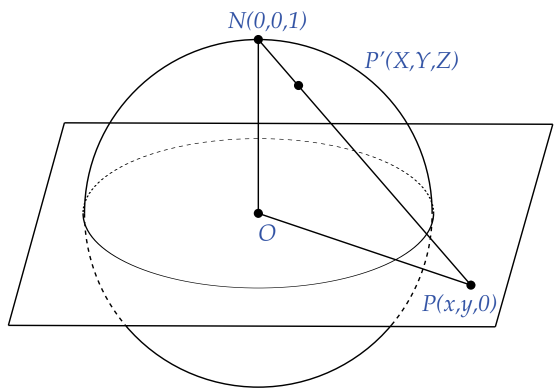

Figure 16.1

Clearly, points for which $|z| < 1$ will map to the lower hemisphere and points for which $|z| > 1$ will map to the upper hemisphere. Also, all points in the complex plane will map to a point of the sphere, covering the sphere entirely, except for N. If we identify the point at infinity with N, we get a one-to-one correspondence between the extended complex plane and the sphere S. The coordinate equations for this map are

$$X = \frac{2x}{|z|^2 + 1}, \; Y = \frac{2y}{|z|^2 + 1}, \; Z = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{16.1}$$

The derivation of these coordinate equations is left as an exercise. These equations define a map from the extended complex plane to the sphere.

Alternatively, we can consider the function π given by $\pi(P') = P$

that maps points on the sphere to points in the complex plane. This map is called the *stereographic projection* of S onto the complex plane.

An important property of stereographic projection is that it maps circles or lines to circles or lines.

Theorem 16.2. *Let c be a circle or line on the unit sphere. Then, the image of c under π is again a circle or line.*

Proof: We note that c is the intersection of some plane with the sphere. Planes have the general equation $AX + BY + CZ = D$, where A, B, C, and D are constants. Then, using equation 16.1 we have

$$A\frac{2x}{|z|^2+1} + B\frac{2y}{|z|^2+1} + C\frac{|z|^2-1}{|z|^2+1} = D$$

Simplifying, we get

$$2Ax + 2By + C(x^2+y^2-1) = D(x^2+y^2+1)$$

or

$$(C-D)x^2 + (C-D)y^2 + 2Ax + 2By = C + D$$

If $C - D = 0$ we get the equation of a line. Otherwise, this is the equation of a circle. □

Stereographic projection also has the property that it preserves *angles*. This is true of any map of the extended complex plane that takes circles and lines to circles and lines (see [4, page 90] or [14, pages 248–254]).

16.1.3 Analytic Functions and Conformal Maps

Two very important properties of a complex function f are its differentiability and its geometric effect on regions in the plane.

Definition 16.5. *A complex function $f(z)$ is* differentiable *at z_0 if*

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. The value of the limit will be denoted as $f'(z_0)$.

The complex derivative of a function satisfies the same rules as for a real derivative: the power rule, product and quotient rules, and the chain rule. However, the fact that the limit defining the derivative is complex yields some interesting differences that one would not expect from comparison with real functions.

For example, the function $f(z) = \overline{z}$ (complex conjugation) is *not* differentiable. To see this, let $z = z_0 + h$. Then

$$\begin{aligned}\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{\overline{z+h} - \overline{z}}{h} = \lim_{h \to 0} \frac{\overline{h}}{h}\end{aligned}$$

If h is real, this limit is 1 and if h is pure imaginary, this limit is -1. Thus, the complex conjugate function is not differentiable.

The functions of most interest to us are those differentiable not only at a point, but in a region about a point.

Definition 16.6. *A function $f(z)$ is* analytic *at a point z_0 if it is differentiable at z_0 and at all points in some small open disk centered at z_0.*

An amazing difference between complex variables and real variables is the fact that an analytic function is not just one-times differentiable, but is in fact infinitely differentiable and has a power series expansion about any point in its domain. The proof of these results would take us far afield of our main focus of study. For a complete derivation of these results on analytic functions, see [17] or [15].

Analytic functions have the geometric property that angles and lengths will *conform* or be in harmony as they are transformed by the action of the function.

Definition 16.7. *Let $f(z)$ be a function defined on an open subset D of the complex plane. Then, we say that*

- f preserves angles *if given two differentiable curves c_1 and c_2 intersecting at z_0 with an angle of θ between their tangents (measured from c_1 to c_2), the composite curves $f \circ c_1$ and $f \circ c_2$ have well-defined tangents that intersect at the same angle θ (measured from $f \circ c_1$ to $f \circ c_2$).*

- f preserves local scale *if for z near z_0, we have $|f(z) - f(z_0)| \approx k|z - z_0|$, with k a positive real constant, and*

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = k$$

Definition 16.8. *A continuous function $f(z)$ defined on an open set D is said to be* conformal *at a point z_0 in D if f preserves angles and preserves local scale.*

It turns out that an analytic function is conformal wherever its derivative is non-zero.

Theorem 16.3. *If a function $f(z)$ is analytic at z_0, and if $f'(z_0) \neq 0$, then f is conformal at z_0.*

Proof: Let $z(t) = c(t)$ be a curve with $c(0) = z_0$. The tangent vector to this curve at $t = 0$ is $c'(0)$, and we can assume this tangent vector is non-zero. (To check conformality, we need to have well-defined angles.) Also, $arg(c'(0))$ measures the angle this tangent makes with the horizontal.

The image of c under f is given by $w(t) = f(c(t))$, and the tangent vector to this curve at $t = 0$ will be $\frac{dw}{dt}$ at $t = 0$. Since

$$\frac{dw}{dt} = f'(c(t))c'(t)$$

the tangent to $w(t)$ at $t = 0$ is $w'(0) = f'(c(0))c'(0) = f'(z_0)c'(0)$. Thus, $w'(0) \neq 0$ and $arg(w'(0)) = arg(f'(z_0)) + arg(c'(0))$. We see that the change in angle between the original tangent to c and the tangent to the image curve w is always $arg(f'(z_0))$. Thus, any two curves meeting at a point will be mapped to a new pair of intersecting curves in such a way that their tangents will both be changed by this constant angle, and thus the angle between the original tangents will be preserved.

For showing preservation of scale, we note that

$$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

Thus, for z_0 close to z, $|f(z)-f(z_0)| \approx |f'(z_0)||z-z_0|$ and f preserves local scale. □

We note here that if f is analytic, then it preserves not only the size of angles, but also their *orientation*, since the angle between two curves is modified by adding $arg(f'(z_0))$ to both tangents to get the new angle between the images of these curves.

The converse to Theorem 16.3 also holds.

Theorem 16.4. *If a function $f(z)$ is conformal in a region D, then f is analytic at $z_0 \in D$, and $f'(z_0) \neq 0$.*

For a proof of this theorem see [1].

The next theorem tells us the nature of a conformal map defined on the complex plane (or the extended complex plane).

Theorem 16.5. *A conformal map f that is one-to-one and onto the complex plane must be of the form $f(z) = az + b$, where $a \neq 0$ and b are complex constants.*

Proof: By the previous theorem we know that f is analytic and thus must have a Taylor series expansion about $z = 0$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$. If the series has only a finite number of terms, then f is a polynomial of some degree n. Then, f' is a polynomial of degree $n-1$, and if it is non-constant, then by the Fundamental Theorem of Algebra, f' must have a zero. But, $f' \neq 0$ anywhere, and thus $f(z) = az + b, a \neq 0$.

Suppose the series for f has an infinite number of terms. Then there are points α in the plane for which $f(z) = \alpha$ has an infinite number of solutions, which contradicts f being one-to-one. (The point α exists by the *Casorati-Weierstrass Theorem* and the fact that $f(\frac{1}{z})$ has an *essential singularity* at $z = 0$ (see [3, page 105] for more details)). □

Theorem 16.6. *A conformal map f that is one-to-one and onto the extended complex plane must be of the form $f(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$.*

Proof: If the point at infinity gets mapped back to itself by f, then f is a conformal map that is one-to-one and onto the regular complex plane; thus $f(z) = az + b, a \neq 0$ by the previous theorem. Then $f(z) = \frac{az+b}{0z+d}$, where $d = 1$ and $ad - bc = a \neq 0$.

Otherwise, suppose that $z = \alpha$ is the point that gets mapped to infinity. Let $\zeta = \frac{1}{z-\alpha}$. Consider $w = f(\zeta)$. At $z = \alpha$ the value of ζ becomes infinite. Thus, for $w = f(\zeta)$, the point at infinity gets mapped to itself, and so $w = f(\zeta) = a\zeta + b, a \neq 0$. Then

$$w = a\left(\frac{1}{z-\alpha}\right) + b = \frac{a + b(z-\alpha)}{z-\alpha} = \frac{bz - (a + b\alpha)}{z-\alpha}$$

Since $-b\alpha - (-(a + b\alpha)(1)) = a \neq 0$, we have proved the result.

□

Definition 16.9. *Functions f of the form $f(z) = \frac{az+b}{cz+d}$ are called* bilinear *transformations, or* linear fractional *transformations. If $ad - bc \neq 0$, then f is called a* Möbius transformation.

We note here that an equivalent definition of Möbius transformations would be the set of $f(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$. (The proof is left as an exercise.)

Exercise 16.1.1. *Derive Equation 16.1. [Hint: If N, P', and P are on a line, then $P' - N$ and $P - N$ are parallel. Use this to show that $(X, Y, Z - 1) = t(x, y, -1)$. Solve this for X, Y, Z and use the fact that $X^2 + Y^2 + Z^2 = 1$ to find t.]*

Exercise 16.1.2. *Show that stereographic projection (the map π described above) has the equation $\pi(P') = \frac{1}{1-Z}(X, Y)$. [Hint: The line through N, P', and P in Figure 16.1 will have the form $N + t(P' - N)$. The third coordinate of points on this line will be given by $1 + t(Z - 1)$. Use this to find t.]*

Exercise 16.1.3. *Show that stereographic projection is a one-to-one map. [Hint: Start with two points (X, Y, Z) and (X', Y', Z') and suppose $\pi(X, Y, Z) = \pi(X', Y', Z')$. Use the previous exercise and Equation 16.1 to show $(X, Y, Z) = (X', Y', Z')$.]*

Exercise 16.1.4. *Show that stereographic projection is onto the complex plane.*

Exercise 16.1.5. *Show that the set of Möbius transformations can be defined as the set of* $f(z) = \frac{az+b}{cz+d}$ *with* $ad - bc = 1$.

Exercise 16.1.6. *Let* $f(z) = \overline{z}$. *Show that* f *has the local scale preserving property, but has the angle-preserving property only up to a switch in the sign of the angle between tangent vectors. Such a map is called* indirectly conformal.