CHAPTER 14

# Foundations of Elliptic Geometry

> It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relationship of these presumptions is left in the dark; one sees neither whether and in how far their connection is necessary, nor a priori whether it is possible.
>
> – Georg Bernhard Riemann (1826–1866) from *On the hypotheses which lie at the foundation of geometry (1854)*

In Chapter 11, we showed that Euclid's Propositions 1-15, 23, and half of 26 (ASA) can be put on a solid axiomatic footing. We used the axioms of betweenness to develop segment and angle ordering, and the idea of separation of points. We also developed the notion of segment and angle measure.

As noted in the section on betweenness (section 11.2 in Chapter 11), the axioms we used to define betweenness do not hold in Elliptic geometry, due to the fact that lines are bounded and thus are forced to loop back on themselves. In this chapter we develop an alternative set of axioms that will serve as a replacement for the betweenness axioms. We also slightly modify the axioms of incidence. Using these new axioms, we will show that almost all of the results from Chapter 11, from sections 11.1 to 11.9, will hold in Elliptic Geometry. The material on

Circle-Circle Continuity in section 11.10 will also hold, as will the results on reflections in section 11.11.

## 14.1 AXIOMS OF INCIDENCE

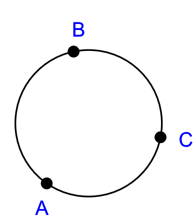
The incidence axioms from section 11.1 will still be valid for Elliptic geometry, but one of these axioms will need to be strengthened. Axiom I-3 stated that "On every line there exist at least two distinct points." We will need to strengthen this to requiring that there exists at least three points on any line. This new condition is necessary for our new axioms on betweenness. Also, we will add the Elliptic Parallel property that all lines intersect, as this is really an incidence property. Thus, the Elliptic Incidence axioms will be:

- **E-I-1 Through any two distinct points $A$ and $B$ there is always a line $m$.**
- **E-I-2 Through any two distinct points $A$ and $B$, there is not more than one line $m$.**
- **E-I-3 On every line there exists at least three distinct points. There exist at least three points not all on the same line.**
- **E-I-4 Through any three points not on the same line, there is one and only one plane.**
- **E-1-5 For any pair of distinct lines $l$ and $m$, there is always a point $P$ that is on both lines.**

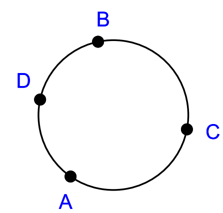
We note here that if these axioms are true in a particular model, then the axioms of section 11.1 will also hold.

## 14.2 AXIOMS OF SEPARATION (BETWEENNESS)

Consider a "line" that loops back on itself. In our models of Elliptic geometry, lines were circles. On a circle, given three points $A$, $B$, and $C$ it is not clear which point is "between" the other two.



However, if we add a fourth point $D$, as shown, it is clear now that $C$ is separated from $D$ by $A$ and $B$. $C$ is "between" $A$ and $B$ relative to point $D$.

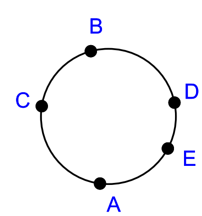



Our new axioms will define this type of separation of points. Our system will follow that of H.S.M. Coxeter in his book *Non-Euclidean Geometry* [6], with some slight modifications. We will use the notation $AB \parallel CD$ to stand for the idea that $A$ and $B$ *separate* $C$ and $D$, or that $C$ is *between* $A$ and $B$ relative to point $D$. As we saw in Chapter 11, we do not define what separation and betweenness mean, we only stipulate the properties these terms have.

- **E-II-1 If $A$, $B$, and $C$ are three collinear points, then there is at least one point $D$ such that $AB \parallel CD$.**
- **E-II-2 If $AB \parallel CD$ then $A$, $B$, $C$, and $D$ are collinear and distinct.**
- **E-II-3 If $AB \parallel CD$ then $AB \parallel DC$ and $CD \parallel AB$.**
- **E-II-4 If $A$, $B$, $C$, and $D$ are four collinear points, then either $AB \parallel CD$ or $AC \parallel BD$ or $AD \parallel BC$.**
- **E-II-5 If $A$, $B$, $C$, $D$, and $E$ are collinear points, and if $AB \parallel CD$ then, either $AB \parallel CE$ or $AB \parallel DE$.**
- **E-II-6 If $AB \parallel CD$ and if there is a perspectivity mapping $A$, $B$, $C$, and $D$ on line $l$ to $A'$, $B'$, $C'$, and $D'$ on line $l'$, then $A'B' \parallel C'D'$.**

The first four axioms seem straight-forward if we keep in mind the circle figure above. The fifth axiom seems a bit strange. Let's consider it in terms of our Elliptic Model.

Here, we have $AB \parallel CD$ and $AB \parallel CE$. It is clear that $D$ and $E$ are not separated from $A$ and $B$.



This figure suggests the following theorem:

**Theorem 14.1.** *Given five distinct collinear points $A$, $B$, $C$, $D$, and $E$, if $AB \parallel CD$ and $AD \parallel BE$, then $AB \parallel CE$, $BE \parallel CD$, and $AD \parallel CE$.*

Proof: Since $AB \parallel CD$, then by axiom E-II-5 we have that either $AB \parallel DE$ or $AB \parallel CE$. By axiom E-II-4 we cannot have both $AD \parallel BE$ and $AB \parallel DE$. Thus, $AB \parallel CE$.

Now, we can assume $AB \parallel CE$. Then, $BA \parallel EC$ by axiom E-II-3. Also, by axiom E-II-3 we have $BE \parallel DA$. Thus, if we let $A' = B$, $B' = E$, $C' = D$, $D' = A$, and $E' = C$, we have $A'B' \parallel C'D'$ and $A'D' \parallel B'E'$. By what we have already proven we must have $A'B' \parallel C'E'$ or $BE \parallel DC$ (equivalently $BE \parallel CD$).

For the last part of the proof, we have shown $EC \parallel AB$ and $EB \parallel CD$. Let $A' = E$, $B' = C$, $C' = A$, $D' = B$, and $E' = D$. Then, $A'B' \parallel C'D'$ and $A'D' \parallel B'E'$ and again we have $A'B' \parallel C'E'$ or $EC \parallel AD$ (equivalently $AD \parallel CE$). □

This theorem could be called "five-point betweenness." It serves the role for elliptic geometry that four-point betweenness served in Chapter 11. This theorem says that, once we know $AB \parallel CD$ and $AD \parallel BE$, then all possible separations illustrated by the relative positions of the five points in Figure 14.1 are true. When we use this result in proofs, we can refer to this figure for insight on five-point betweenness.

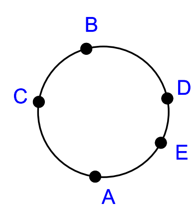



Figure 14.1 Five-Point Betweenness

Axiom E-II-6 deals with perspectivities.

**Definition 14.1.** *A* perspectivity with center *$O$ is a 1-1 mapping of the points of line $l$ to the points of line $l'$ such that if $A$ on $l$ is mapped to $A'$ on $l'$, then $\overleftrightarrow{AA'}$ passes through $O$. Also, $O$ is not on either line (Figure 14.2).*

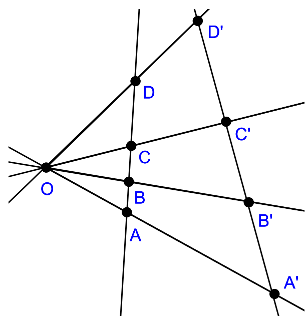



Figure 14.2 Perspectivity

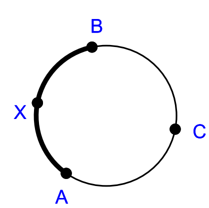
One can think of the point $O$ as a light source and the points on $l'$ as the shadows of the projection from points on $l$. Axiom E-II-6 holds true in all of our models of Euclidean, Hyperbolic, Elliptic, and Projective geometries.

We will use the axioms of separation to define segments and (relative) betweenness.

**Definition 14.2.** *If $A$, $B$, and $C$ are three collinear points, then the* segment *$\overline{AB}/C$ is the set of points $X$ such that $AB \parallel CX$, together with points $A$ and $B$.*

We can think of a segment $\overline{AB}/C$ as the set of points on $\overleftrightarrow{AB}$ that are separated from $C$ by $A$ and $B$, together with endpoints $A$ and $B$.

In our Elliptic Model, this definition allows us to specify one of the two possible segments defined by $A$ and $B$.



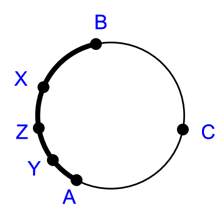

In Euclidean and Hyperbolic geometry, two points determine a unique segment. In Elliptic geometry the best we can do is the following:

**Theorem 14.2.** *Given two distinct points $A$ and $B$ there is at least one segment containing $A$ and $B$.*

Proof: The proof of this theorem is essentially an application of the Incidence Axioms for Elliptic geometry, along with Axiom E-II-1, and is left as an exercise. □

**Definition 14.3.** *$X$ is an* interior point *of $\overline{AB}/C$ if $X$ is an element of $\overline{AB}/C$ that is not equal to $A$ or $B$. An interior point $Z$ is* between *two interior points $X$ and $Y$ of $\overline{AB}/C$ if $XY \parallel CZ$. We then say that $X * Z * Y/C$. A point $Y$ is an* exterior point *of $\overline{AB}/C$ if it is not $A$ or $B$ or an interior point.*

If we restrict points to only those interior to a specified segment, we can refer to "three-point" betweenness without ambiguity. Here, interior point $Z$ is between $X$ and $Z$. It will be left as an exercise to show that this "three-point" betweenness satisfies axioms II-1 through II-3 of section 11.2.




**Theorem 14.3.** *Given a segment $\overline{AB}/C$ there is at least one interior point.*

Proof: The proof of this theorem is essentially an application of Axiom E-II-1 and is left as an exercise. □

In the figures above, which are motivated by our Elliptic Model, it appears that, given a segment, there is a "complimentary" segment. The existence of interior points allows us to define complimentary segments precisely.

**Definition 14.4.** *Given a segment $\overline{AB}/C$, let $D$ be an interior point. Then, the segment $\overline{AB}/D$ is called the* complimentary segment *and will be denoted by* $\underset{C}{\overline{AB}}$

We note that $X$ is an interior point of $\underset{C}{\overline{AB}}$, with $D$ an interior point to $\overline{AB}/C$, if and only if $AB \parallel DX$. This follows directly from the definition of segment $\overline{AB}/D$. Since $AB \parallel DC$, it is always the case that $C$ is an interior point of $\overline{AB}/D$, thus also of $\underset{C}{\overline{AB}}$. The next theorem is useful for telling when two points are interior to a segment.

**Theorem 14.4.** *Let $X$ be an interior point of $\overline{AB}/C$. Then, $Y$ on $\overleftrightarrow{AB}$ is also an interior point if and only if $AX \parallel BY$ or $AY \parallel BX$.*

Proof: Let $X$ be an interior point of $\overline{AB}/C$. Then, $AB \parallel CX$.

Assume $Y$ is also an interior point of $\overline{AB}/C$. Then $AB \parallel CY$. Suppose $AB \parallel XY$. This is impossible by Axiom E-II-5, as we cannot have $AB \parallel CX$, $AB \parallel CY$, and $AB \parallel XY$ simultaneously. So, $AB \nparallel XY$. By axiom E-II-4 either $AX \parallel BY$ or $AY \parallel BX$.

Conversely, suppose $AX \parallel BY$ or $AY \parallel BX$. If $AX \parallel BY$, then since $AB \parallel CX$, we have by Theorem 14.1 that $AB \parallel CY$ and $Y$ is interior to $\overline{AB}/C$. The proof of the remaining case of $AY \parallel BX$ is similar. □

An (almost) immediate corollary to this theorem is the following:

**Corollary 14.5.** *Let $X$ be an interior point of $\overline{AB}/C$. Then, $Y$ on $\overleftrightarrow{AB}$ is an* exterior *point if and only if $AB \parallel XY$.*

This result implies that if $Y$ is an exterior point of $\overline{AB}/C$, then $\overline{AB}/C = \overline{AB}/Y$.

The next theorem guarantees that we can split a segment into two distinct parts.

**Theorem 14.6.** *Let $X$ be an interior point of $\overline{AB}/C$. Then $\overline{AB}/C = \overline{AX}/C \cup \overline{XB}/C$ and $X$ is the only point common to $\overline{AX}/C$ and $\overline{XB}/C$.*

Proof: We will first show that $\overline{AB}/C \subset \overline{AX}/C \cup \overline{XB}/C$. Clearly, $X$ is in either $\overline{AX}/C$ or $\overline{XB}/C$. Let $Y \neq X$ be an interior point on $\overline{AB}/C$. By Theorem 14.4 we know that $AX \parallel BY$ or $AY \parallel BX$. Suppose $AX \parallel BY$. Since $Y$ is on $\overline{AB}/C$ we have that $AB \parallel YC$. Then, by five-point betweenness (Theorem 14.1) we have that $AX \parallel CY$ and thus $Y$ is on $\overline{AX}/C$. A similar argument shows that if $AY \parallel BX$ then $Y$ is on $\overline{XB}/C$.

Now, we will show that $\overline{AX}/C \cup \overline{XB}/C \subset \overline{AB}/C$. We know that $X$ is on $\overline{AB}/C$, by the hypothesis of the theorem. Let $Y$ be an interior point of $\overline{AX}/C$. Then, $AX \parallel CY$. Since $X$ is on $\overline{AB}/C$ we know that $AB \parallel XC$. By five-point betweenness, we have $AB \parallel CY$, and $Y$ is on $\overline{AB}/C$. A similar argument shows that if $Y$ is interior to $\overline{XB}/C$, then $Y$ is on $\overline{AB}/C$.

The proof that there is only one point common to $\overline{AX}/C$ and $\overline{XB}/C$ is left as an exercise. □

The next theorem tells us that the definition of the complimentary segment $\underset{C}{\overline{AB}}$ to a given segment $\overline{AB}/C$ does not depend on the choice of interior point $D$ on $\overline{AB}/C$. It also implies that lines are "finite" in the sense that they are composed of two segments. (Note: This does not mean they have finite length, as we have not defined length yet.)

**Theorem 14.7.** *Given a line $\overleftrightarrow{AB}$ and a segment $\overline{AB}/C$ then:*

- $\overline{AB}/C \cap \underset{C}{\overline{AB}} = \{A, B\}$,
- $\overline{AB}/C \cup \underset{C}{\overline{AB}} = \overleftrightarrow{AB}$.

Proof:

- Clearly, by using the definitions of $\overline{AB}/C$ and $\underset{C}{\overline{AB}}$ we have that $\{A, B\} \subset \overline{AB}/C \cap \underset{C}{\overline{AB}}$.

  On the other hand let $X$ be an element of $\overline{AB}/C \cap \underset{C}{\overline{AB}}$. If $X$ is $A$ or $B$ then $X$ is clearly in $\{A, B\}$. Assume $X$ is not $A$ and not $B$. Since $X$ is in $\overline{AB}/C$ we have that $AB \parallel CX$. Since $X$ is in $\underset{C}{\overline{AB}}$, then $AB \parallel DX$ for $D$ an interior point of $\overline{AB}/C$. But, we know that $AB \parallel CD$. By axiom E-II-5, all three of $AB \parallel CX$, $AB \parallel DX$, and $AB \parallel CD$ cannot hold simultaneously. Thus, there is no point common to the interiors of $\overline{AB}/C$ and $\underset{C}{\overline{AB}}$ and so $\overline{AB}/C \cap \underset{C}{\overline{AB}} \subset \{A, B\}$.

Thus, $\overline{AB}/C \cap \underset{C}{\overline{AB}} = \{A, B\}$.

- Let $X$ be an element of $\overline{AB}/C \cup \underset{C}{\overline{AB}}$. If $X = A$ or $X = B$ then, $X$ is certainly on the line $\overleftrightarrow{AB}$. Otherwise, suppose $X$ is on $\underset{C}{\overline{AB}}$ with $X$ not equal to $A$ or $B$. Then, $AB \parallel DX$ for some interior point $D$ of $\overline{AB}/C$. But, this implies by axiom E-II-2 that $A, B, D, X$ are collinear. Likewise, if $X$ is on $\overline{AB}/C$, then $A, B, C, X$ are collinear. In either case, we get that $X$ is on $\overleftrightarrow{AB}$.

  On the other hand let $X$ be an element of $\overleftrightarrow{AB}$. By axiom E-II-4 we have that either $AB \parallel CX$, or $AC \parallel BX$, or $AX \parallel BC$. If $X = A$ or $X = B$ or $AB \parallel CX$, then $X$ is on $\overline{AB}/C$. If $AC \parallel BX$, then since $AB \parallel CD$, for $D$ interior to $\overline{AB}/C$, we have by Theorem 14.1 that $AB \parallel DX$. Thus, $X$ is on $\underset{C}{\overline{AB}}$. If $AX \parallel BC$, then since $AB \parallel CD$, we can use axiom E-II-3 to rewrite these two separations as $BA \parallel CD$ and $BC \parallel AX$. Again, by Theorem 14.1, we have $BA \parallel DX$ and $X$ is on $\underset{C}{\overline{AB}}$.

  We have shown that $\overline{AB}/C \cup \underset{C}{\overline{AB}} \subset \overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \subset \overline{AB}/C \cup \underset{C}{\overline{AB}}$. Thus, $\overline{AB}/C \cup \underset{C}{\overline{AB}} = \overleftrightarrow{AB}$.

□

Given two points $A$ and $B$, we then have two possible segments that are defined by these points. When we speak of a "choice" for segment $\overline{AB}$ we will be referring to a selection of one of the two possible segments defined by $A$ and $B$. With this understanding, Pasch's axiom for Elliptic geometry becomes a theorem:

**Theorem 14.8.** *(Pasch's Axiom) Let $A$, $B$, and $C$ be three non-collinear points and let $m$ be a line that does not contain any of these points. If $m$ contains a point of segment $\overline{AB}/E$, for some $E$ on $\overleftrightarrow{AB}$ then, for a given choice of $\overline{AC}$, there is a unique choice of $\overline{BC}$ such that $m$ contains a point of either $\overline{AC}$ or $\overline{BC}$.*

Proof: The proof is left as an exercise. □

This theorem has a conclusion that seems rather wishy-washy. This is due to the fact that our Euclidean notion of betweenness does not apply, so we cannot specify which segments on a line will be intersected. The next theorem is more satisfying in this respect. It says that whenever a pair of lines intersect a choice of a side on a triangle, and do not intersect another side, then both lines intersect the same choice of the third side.

**Theorem 14.9.** *Let $A$, $B$, and $C$ be three non-collinear points and let $m$, $n$ be lines that do not contain any of these points. For a choice of $\overline{AC}$, suppose that $m$ and $n$ do not intersect $\overline{AC}$. Let $\overline{AB}/E$ be a choice of segment $\overline{AB}$ and $\underset{E}{AB}$ be the complimentary segment. Then,*

- *If $m$ and $n$ intersect $\overline{AB}/E$, then there is a unique choice of $\overline{BC}$, say $\overline{BC}/F$, such that $m$ and $n$ both intersect $\overline{BC}/F$.*
- *If $m$ intersects $\overline{AB}/E$ and $n$ intersects $\underset{E}{AB}$ then there is a unique choice of $\overline{BC}$, say $\overline{BC}/F$, such that $m$ intersects $\overline{BC}/F$ and $n$ intersects $\underset{F}{BC}$.*

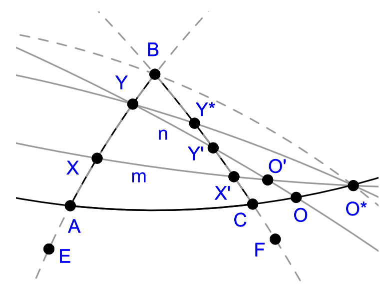
Proof: Let $m$ and $n$ intersect $\overline{AB}/E$ at $X$ and $Y$. Then, $X$ and $Y$ are internal to $\overline{AB}/E$, and by Theorem 14.4, we have that $AX \parallel BY$ or $AY \parallel BX$. Now, $m$ intersects some choice of $\overline{BC}$ by the previous theorem. Without loss of generality, we assume it intersects $\overline{BC}/F$ at $X'$. Also, $n$ intersects some choice of $\overline{BC}$ at a point $Y'$.

By Axiom E-1-5, $m$ and $\overleftrightarrow{AC}$ intersect at some point $O^*$, and since $O^*$ is not any of the points $A$, $B$, or $C$, and since $m$ cannot coincide with either $\overleftrightarrow{AB}$ or $\overleftrightarrow{BC}$, we get that $O^*$ is not on $\overleftrightarrow{AB}$ or $\overleftrightarrow{BC}$. Thus, we get a perspectivity from $O^*$ mapping $\overleftrightarrow{AB}$ to $\overleftrightarrow{BC}$. We also note that the hypotheses of the theorem imply that $O^*$ is external to the given choice of $\overline{AC}$. Let $Y^*$ be the intersection of $\overleftrightarrow{YO^*}$ with $\overleftrightarrow{BC}$.

By axiom E-II-6 we know that perspectivities preserve separation. Thus, since $AX \parallel BY$ or $AY \parallel BX$, we have that $CX' \parallel BY^*$ or $CY^* \parallel BX'$. Since $X'$ is internal to $\overline{BC}/F$, we have that $Y^*$ is also internal, by Theorem 14.4.

Now, $m$ and $n$ will intersect at a point $O'$. Also, $n$ will intersect $\overleftrightarrow{AC}$ at a point $O$ with $O$ external to the given choice of $\overline{AC}$.

Here is an illustration of what we have developed so far, as it might appear in the circle model of Elliptic geometry. Note that we must be careful not to make conclusions based on this figure, but use it only as a visual aid.




Suppose $O = O^*$, then $O' = O^*$ and we have that $Y' = Y^*$. Thus $CX' \parallel BY'$ or $CY' \parallel BX'$. By Theorem 14.4, we have that $X'$ and $Y'$ are interior to $\overline{BC}/F$.

Suppose $O \neq O^*$. Since two external points of a segment are internal points of the complimentary segment, we have by Theorem 14.4 that $CO \parallel AO^*$ or $CO^* \parallel AO$. Then, since $Y$ is not on $\overleftrightarrow{AC}$ or $\overleftrightarrow{BC}$, we have a perspectivity defined from $Y$ mapping $\overleftrightarrow{AC}$ to $\overleftrightarrow{BC}$. Under this perspectivity, $A$ goes to $B$, $C$ goes to itself, $O$ goes to $Y'$, and $O^*$ goes to $Y^*$. By axiom E-II-6, we have $CY' \parallel BY^*$ or $CY^* \parallel BY'$. Since $Y^*$ is internal to $\overline{BC}/F$, we have by Theorem 14.4, that $Y^*$ and $Y'$ are internal to $\overline{BC}/F$. We conclude that $X'$ and $Y'$ are interior to $\overline{BC}/F$.

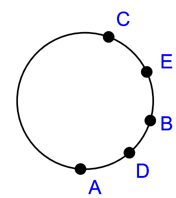
The proof of the second part of the theorem follows immediately by the properties of segments and their compliments. □

The next result says that we can always embed a segment in a "larger" segment. This is as close as we can get to the usual Euclidean (and Hyperbolic) notion that segments can always be extended.

**Theorem 14.10.** *Given a segment $\overline{AB}/C$ we can always find two points $E \neq F$ on $\overleftrightarrow{AB}$, exterior to $\overline{AB}/C$, such that $\overline{AB}/C$ is contained in $\overline{EF}/C$.*

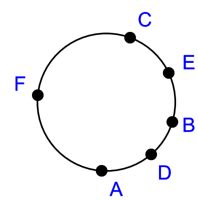
Proof: By Theorem 14.3, $\overline{AB}/C$ has an interior point $D$. Also, $\overline{BC}/A$ has an interior point $E$. Then $BC \parallel AE$ and $E$ cannot be one of $A$, $B$, or $C$. If $E = D$, then $BC \parallel AD$, which contradicts $DC \parallel AB$, by axiom E-II-4. Since $D$ could be any interior point, we have that $E$ must be exterior to $\overline{AB}/C$. By Corollary 14.5, $AB \parallel DE$.

Consider segment $\underset{E}{\overline{AC}}$. $E$ is an interior point to this segment. Since $AE \parallel CB$ and $AB \parallel ED$, then by Theorem 14.1 we have that $AE \parallel CD$. By Theorem 14.4, $D$ is an interior point of $\underset{E}{\overline{AC}}$.




Since $D$ is an arbitrary interior point of $\overline{AB}/C$, then the interior of $\overline{AB}/C$ is contained in $\underset{E}{\overline{AC}}$. Also, since $AE \parallel BC$, then $B$ is interior to $\underset{E}{\overline{AC}}$ by Theorem 14.4. Thus, $\overline{AB}/C$ is contained in $\underset{E}{\overline{AC}}$.

Let $F$ be an interior point of $\overline{AC}/E$. Then $AC \parallel EF$, and so $A$ is on $\overline{EF}/C$. Since $B$ and $D$ are also interior to $\underset{E}{\overline{AC}} = \overline{AC}/F$, then $AC \parallel BF$ and $AC \parallel DF$.




Since $AE \parallel BC$ and $AC \parallel EF$, then by Theorem 14.1 we have $AE \parallel BF$ and, since $A$ is already interior to $\overline{EF}/C$, then by Theorem 14.4, $B$ is also interior to $\overline{EF}/C$. Likewise, since $AE \parallel CD$ and $AC \parallel FE$, then $AE \parallel FD$ and $D$ is interior to $\overline{EF}/C$. We conclude that $\overline{AB}/C$ is contained in $\overline{EF}/C$.

Finally, we must show $F$ is exterior to $\overline{AB}/C$. Since $F$ is interior to $\overline{AC}/E$, then $F \neq A$. If $F = B$, we would have $AC \parallel EB$. But this is impossible, as we have shown that $AE \parallel CB$ and we cannot have both $AC \parallel EB$ and $AE \parallel CB$ by axiom E-II-4. If $F = D$ (an arbitrary interior point), we would have $AC \parallel ED$. But, we have shown that $AE \parallel CD$, so $AC \nparallel ED$ by axiom E-II-4. Thus, $F$ must be exterior to $\overline{AB}/C$. $\square$

**Exercise 14.2.1.** *Show that "three-point betweenness" from definition 14.3 satisfies axioms II-1 through II-3 of section 11.2. [Note: in axioms II-1 to II-3 you will need to substitute "segment" for "line".]*

**Exercise 14.2.2.** *Prove Theorem 14.2.*

**Exercise 14.2.3.** *Prove Theorem 14.3.*

**Exercise 14.2.4.** *Prove Theorem 14.8. [Hint: Use Theorem 14.7 and the fact that every pair of distinct lines intersects.]*

**Exercise 14.2.5.** *Finish the uniqueness part of the proof of Theorem 14.6. [Hint: suppose there were two points common to the intersection of $\overline{AX}/C$ and $\overline{XB}/C$. Find a contradiction to axiom E-II-5.]*

## 14.3 ORIENTATION, RAYS, AND ANGLES

From segments the next logical geometric concept to define would be *rays*. However, this will be a bit tricky, as we cannot assume that a point on a line divides the line into two separate pieces. What is needed is an idea of *orientation* from a point on a line, so we can talk about opposite directions from a given point. The development we give below is that of H.M.S. Coxeter in [6][Chapter 2]. We start with three points on a line.

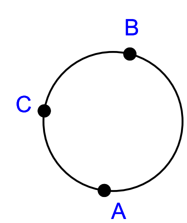
**Theorem 14.11.** *Let $A$, $B$, and $C$ be three points on a line. Then,*

- *the union of the segments $\overline{AB}/C$, $\overline{BC}/A$, and $\overline{CA}/B$ is the entire line*
- *the pair-wise intersection of these segments contains only their endpoints.*

Proof:

- Let $X$ be a point on the line that is not one of $A$, $B$, or $C$. Then, by axiom E-II-4 we have that either $AB \parallel CX$ or $AC \parallel BX$ or $BC \parallel AX$. By the definition of segments, either $X$ is on $\overline{AB}/C$ or $\overline{CA}/B$ or $\overline{BC}/A$. On the other hand, if $X$ is on $\overline{AB}/C$ or $\overline{BC}/A$ or $\overline{CA}/B$ then it is on the line by Theorem 14.7 .
- Suppose $X$ is on both $\overline{AB}/C$ and $\overline{CA}/B$. If $X$ is not one of the endpoints, then $AB \parallel CX$ and $AC \parallel BX$. This is impossible by axiom E-II-4. The proof for the other two pairings is similar. □

In the Elliptic Model, the result of this theorem is quite clear. The three points divide the "line" into three parts.

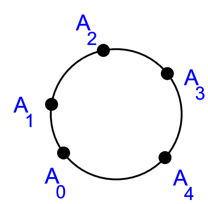



We can extend the preceding result to any set of $n$ points on a line.

**Theorem 14.12.** *Given $n$ distinct points on a line, there is a labeling of the points $A_0, A_1, \ldots A_{n-1}$ such that the union of the segments $\overline{A_r A_{r+1}}/A_{r-1}$, for $r = 0, \ldots n-1$ is the entire line (with indices computed mod $n$), and the pair-wise intersections of segments contains only endpoints.*

Proof: The proof is by induction on $n$. The case of $n = 2$ or $n = 3$ was shown above. Suppose the results hold for $n - 1$ where $n > 3$. We will show it holds for $n$. Of the $n$ points, choose a subset of $n-1$. We can label these $A_0, A_1, \ldots A_{n-2}$ such that the union of the segments $\overline{A_r A_{r+1}}/A_{r-1}$, for $r = 0, \ldots n-2$ is the entire line (with indices computed mod $n-1$), and the pair-wise intersections of segments containing only endpoints. Let $X$ be the *nth* point. Clearly, $X$ must be an interior point to one of the segments $\overline{A_r A_{r+1}}/A_{r-1}$. Assume it is in $\overline{A_1 A_2}/A_0$. Then, $A_1 A_2 \parallel A_0 X$. As in the proof of the previous theorem, $X$ will divide $\overline{A_1 A_2}/A_0$ into two parts $\overline{A_1 X}/A_0$ and $\overline{X A_2}/A_1$ By re-labeling $X$ to be $A_2$ and shifting the index of $A_2, A_3, \ldots, A_{n-2}$ up one unit, the result is true for $n$ points. □

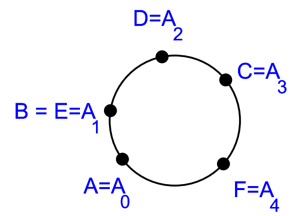
Here is an example in the Elliptic Model of five points dividing a line.




We can now define an orientation based on triples of points.

**Definition 14.5.** *Let $ABC$ and $DEF$ be two triples of points on a line. Then, the distinct points of this set (some points in the first triple may coincide with points of the second triple) can be labeled $A_0, A_1, \ldots, A_{n-1}$ where $n = 3, 4, 5,$ or $6$ as in accordance with the previous theorem. We can assume the labeling is such that $A = A_0$, $B = A_b$, and $C = A_c$, where $b < c$. Then, $D = A_d$, $E = A_e$, and $F = A_f$. If $d < e < f$ or $e < f < d$ or $f < d < e$ we say that the two triples have the* same orientation. *Otherwise, we say the triples have* opposite orientations.

Here is an example in the circle model of Elliptic geometry. $ABC$ has indices $0 < 1 < 3$ but $DEF$ has indices $2 > 1$ and $1 < 4$. These two triples have *opposite* orientations.



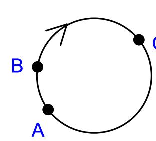

We can now define what we mean by a ray. Intuitively, a ray should be the set of points on a segment, say $\overline{AB}/C$, together with points $D$ outside of the segment that are in the same direction as points $A$, $B$, and $C$.

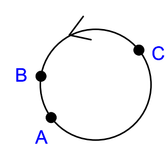
> **Definition 14.6.** *The* ray $\overrightarrow{AB}/C$ *from point $A$ to point $B$ (relative to $C$) is the set of points on $\overline{AB}/C$ together with points $D$ that are exterior to $\overline{AB}/C$ on $\overleftrightarrow{AB}$ such that $ABC$ and $ABD$ have the same orientation.*

In the exercises it is shown that $ABC$ and $ABD$ have the same orientation if and only if $AB$ does not separate $CD$. Thus, points $D$ on $\overrightarrow{AB}/C$ that are not on $\overline{AB}/C$ are by definition in the complimentary segment $\underset{C}{\overline{AB}}$. This means (using Theorem 14.7) that a ray $\overrightarrow{AB}/C$ coincides with the entire line $\overleftrightarrow{AB}$. However, since $ABC$ and $ACB$ have different orientations, we can still define *opposite* rays from a point.

> **Definition 14.7.** *Let $A$, $B$, and $C$ be points on a line $l$. Then, rays $\overrightarrow{AB}/C$ and $\overrightarrow{AC}/B$ are called* opposite rays *from $A$ on $l$.*

Here we have the two oriented rays from $A$ illustrated by arrows in our Elliptic Model:




We can now define an angle in Elliptic geometry.

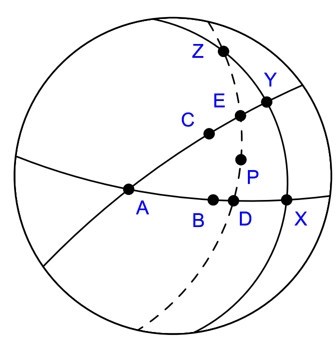
**Definition 14.8.** *Let $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$ be two oriented rays not lying on the same line. This pair of rays is called an* angle*, the rays are called the* sides *of the angle, and $A$ is called the* vertex*.*

We can also define the *interior* of an angle.

**Definition 14.9.** *Let $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$ be two sides of an angle and let $\overline{XY}/Z$ be a segment with exterior point $Z$. A point $P$ is called an* interior point *of the angle (relative to $Z$) if it is not an element of one of the sides and one of the choices for $\overline{PZ}$ intersects exactly one side of the angle. A point that is not in the interior, and not on one of the sides, will be called an* exterior point *of the angle. $Z$ will be an exterior point.*

We note here that this definition implies that an exterior point $R$ will have the property that one choice of $\overline{RZ}$ will intersect neither side of the angle and the other choice of $\overline{RZ}$ will intersect both sides. This is due to the fact that a ray is also a line and every pair of lines intersects. Also, for an interior point $P$, one choice of $\overline{PZ}$ will intersect one side of the angle, which means the other choice must intersect the other side.

Here, we have an illustration of this definition in the circle model of Elliptic geometry. Point $P$ is an interior point of $\angle BAC$. One choice of $\overline{PZ}$ intersects $\overrightarrow{AB}/X$ at $D$ and the other choice intersects $\overrightarrow{AC}/Y$ at $E$.




There is a duality to the definition of interior and exterior points in the definition of an angle. The "choice" of $\overline{XY}/Z$ as having exterior point $Z$ is completely arbitrary. We could just as well have chosen the complimentary segment $\overline{XY}/Q$ where $Q$ was an interior point to $\overline{XY}/Z$. This choice of segment would flip the roles of exterior and interior points of the angle. So, there are two possible choices for an angle just as there were two choices for a segment.

It can also be shown, using the definition of an angle, that an angle divides the points in the Elliptic plane into three disjoint groups – those on one of the sides of the angle, those interior, and those exterior (exercise).

Since the definition of interior points of angles involves interior points of segments, and since interior points of segments are defined by separation, then interior points of angles can be characterized by separation, much like they are in Euclidean geometry.

**Theorem 14.13.** *Let $\angle BAC$ be an angle with vertex $A$. Let the interior of the angle be defined with respect to a given exterior point $Z$. Let $P \neq A$ be a point and let $\overleftrightarrow{PZ}$ intersect $\overrightarrow{AB}/X$ at $D$ and $\overrightarrow{AC}/Y$ at $E$. Then, $P$ is an interior point if and only if $PZ \parallel DE$.*

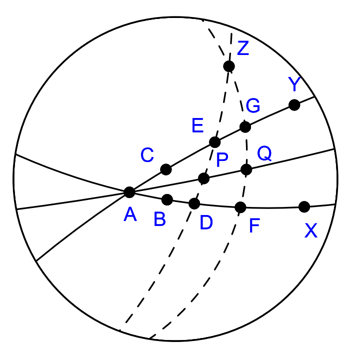
Proof: The proof relies on the definition of interior points of angles and the defintion of segments in terms of separation. This is a good review of separation concepts and is left as an exercise. □

The next theorem will be useful in analyzing interior points of an angle.

**Theorem 14.14.** *Let $\angle BAC$ be an angle with vertex $A$. Let $P$ be an interior point (with respect to a given exterior point $Z$). Then, all points $Q \neq A$ on $\overleftrightarrow{AP}$ are interior points of the angle.*

Proof:

Since $P$ is an interior point of the angle, then there is a choice of $\overline{PZ}$ that intersects $\overrightarrow{AB}/X$ at $D$ and the complimentary segment intersects $\overrightarrow{AC}/Y$ at $E$. By the previous theorem we have $PZ \parallel DE$.



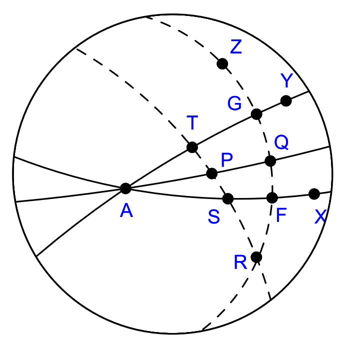
We know that $\overleftrightarrow{QZ}$ intersects $\overrightarrow{AB}/X$ at some point $F$ and intersects $\overrightarrow{AC}/Y$ at some point $G$. Using Axiom E-II-6, and the perspectivity from point $A$, we see that the separation $PZ \parallel DE$ must be preserved on $\overleftrightarrow{QZ}$. That is, $QZ \parallel FG$. By the previous theorem we have that $Q$ is interior to the angle. □

The next theorem tells us how separation can be used to differentiate interior versus exterior points in an angle.

**Theorem 14.15.** *Let $\angle BAC$ be an angle with vertex $A$. Let $P$ be an interior point (with respect to a given exterior point $Z$). Let $T \neq A$ be a point on $\overrightarrow{AC}/Y$. Let $S$ be the intersection of $\overleftrightarrow{PT}$ with $\overrightarrow{AB}/X$. If $R$ is a point on $\overleftrightarrow{PT}$ with $ST \parallel PR$, then $R$ is exterior to the angle.*

Proof: It is clear that $S \neq A$ as $P$ is not on $\overrightarrow{AC}/Y$. Likewise, $R \neq A$. Consider $\overleftrightarrow{RZ}$. If this line passes through $A$, then, $R$ must be an exterior point, because if it was interior, then by the preceding theorem, $Z$ would be an interior point, which it is not.

If $\overleftrightarrow{RZ}$ does not pass through $A$, then it will intersect $\overrightarrow{AB}/X$ at some point $F$, $\overrightarrow{AC}/Y$ at some point $G$ and $\overleftrightarrow{AP}$ at some point $Q$. Since $Q$ is on $\overleftrightarrow{AP}$ then by the previous theorem $Q$ is an interior point. By Theorem 14.13, we have $QZ \parallel FG$.




Using Axiom E-II-6, and the perspectivity from point $A$, we see that the separation $ST \parallel PR$ must be preserved on the perspectively mapped points on $\overleftrightarrow{RZ}$. Thus, we have $FG \parallel QR$. Then, since $FG \parallel QR$ and $FG \parallel QZ$, then by Axiom E-II-5, we know that $FG \nparallel RZ$. This says that $R$ cannot be interior to the angle, by Theorem 14.13. □

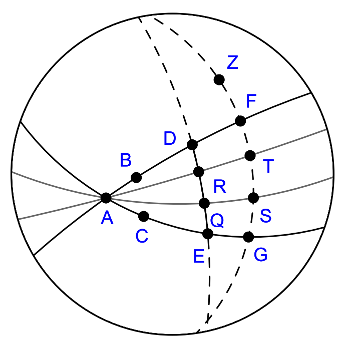
In Chapter 11 we showed that for a non-elliptic angle, if you consider two points on the sides of that angle, then the segment defined by those points has the property that interior points of the segment are interior points of the angle. The next result is directly analogous to this property.

**Theorem 14.16.** *Given an angle $\angle BAC$ suppose $D$ and $E$ are points (distinct from $A$) on the two different sides of the angle. Then there is a choice of $\overline{DE}$ such that all points interior to that segment are interior points of the angle.*

Proof: First, we show that there is a choice of $\overline{DE}$ that has an interior point of the angle. We know that $\overleftrightarrow{DZ}$ will intersect $\overleftrightarrow{AC}$ at some point $W \neq A$, as the two sides are on different lines and $Z$ is not on a side. By Theorem 14.3, we know that $\overline{BW}/Z$ has an interior point $P$. Then, $BW \parallel ZP$, and $P$ is interior to the angle by the definition of interior points. We know that $\overleftrightarrow{AP}$ will intersect $\overleftrightarrow{DE}$ at some point $Q$, and $Q$ is on a choice of $\overline{DE}$. By Theorem 14.14, $Q$ is interior to the angle.

With $Q$ on the given choice of $\overline{DE}$, suppose $R$ is another interior point of the segment.

Then, by Theorem 14.4 we have $DQ \parallel ER$ or $DE \parallel EQ$. Without loss of generality we assume $DQ \parallel ER$. Since every line has at least three points, then on $\overrightarrow{AQ}$ there is a point $S$ distinct from $A$ and $Q$. By Theorem 14.14, $S$ is interior to the angle. Also, $S$ is not on our choice of $\overline{DE}$, as the sides of the angle are not collinear.



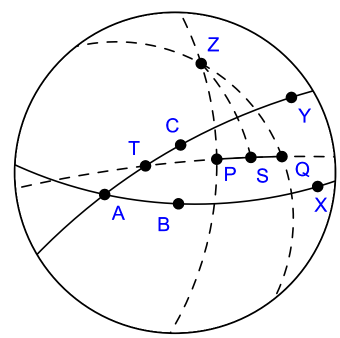

Now, let $\overleftrightarrow{ZS}$ intersect the sides of the angle at $F$ and $G$. We note that $\overleftrightarrow{DE}$ and $\overleftrightarrow{ZS}$ are distinct lines. Since $S$ is interior to the angle, we have $ZS \parallel FG$ by Theorem 14.13. Let $T$ be the intersection of $\overleftrightarrow{AR}$ with $\overleftrightarrow{ZS}$. The perspectivity from $A$ will map $D$ to $F$, $R$ to $T$, $Q$ to $S$ and $E$ to $G$. Since $DQ \parallel ER$, then by axiom E-II-6, we have $FS \parallel GT$. Since $FS \parallel TG$ and $FG \parallel SZ$, we have by five-point betweenness that $FS \parallel TZ$, and $T$ must be interior to the angle by Theorem 14.13. Since $R$ is on $\overleftrightarrow{AT}$, then $R$ is interior to the angle, by Theorem 14.14. □

The next lemma will be useful in showing that interior points form *connected* sets.

**Lemma 14.17.** *Let $\angle BAC$ be an angle with vertex $A$. Let $P$ and $Q$ be interior points (with respect to a given exterior point $Z$). Then, there is a choice of $\overline{PQ}$ such that all points on that choice do not lie on the sides of the angle.*

Proof:

Since every pair of lines intersects, then there is a choice of $\overline{PQ}$, say $\overline{PQ}/S$ for $S$ on $\overleftrightarrow{PQ}$ such that $\overline{PQ}/S$ intersects $\overrightarrow{AC}/Y$ at some point $T$. Then, the complimentary segment $\overline{PQ}/T$ does not intersect $\overrightarrow{AC}/Y$. We note that $S$ can be considered an arbitrary point on $\overline{PQ}/T$ and that $ST \parallel PQ$.




Now, suppose that $S$ is on $\overrightarrow{AB}/X$. Then, by the preceding theorem, $Q$ would have to be an exterior point. Since $Q$ is not exterior, then $S$ is not on $\overrightarrow{AB}/X$. $\square$

It follows from this result that the interior of an angle is a *connected* region.

**Theorem 14.18.** *Let $\angle BAC$ be an angle with vertex $A$. Let $P$ and $Q$ be interior points (with respect to a given exterior point $Z$). Then, there is a choice of $\overline{PQ}$ such that all points on that choice are interior points to the angle.*

Proof: In the proof of the preceding theorem, for the given choice of $\overline{PQ}$, and for a point $S$ on $\overline{PQ}$, consider the triple $PSZ$. $\overrightarrow{AC}/Y$ intersects a choice of $\overline{PZ}$ and $\overrightarrow{AB}/X$ intersects the complimentary segment to $\overline{PZ}$. By the preceding theorem we know that $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$ do not intersect $\overline{PS}$. We know that $\overrightarrow{AC}/Y$ intersects a choice of $\overline{SZ}$ (axiom E-1-5). By Theorem 14.9, $\overrightarrow{AB}/X$ intersects the complimentary choice. Thus, $S$ is an interior point to the angle. $\square$

We are now in a position to define triangles.

**Definition 14.10.** *Given three non-collinear points $A$, $B$, and $C$, the* triangle $\Delta ABC$ *is the set of points on a choice of the three segments $\overline{AB}/X$, $\overline{BC}/Y$, and $\overline{CA/Z}$. These segments are called the* sides *of the triangle.*

**Definition 14.11.** *A point is in the* interior *of triangle $\Delta ABC$ if it is in the intersection of the interiors of its angles $\angle CAB$, $\angle ABC$, $\angle BCA$ which are defined by rays $\overrightarrow{AB}/X$, $\overrightarrow{BC}/Y$, and $\overrightarrow{CA/Z}$ (rays defined by the sides of the triangle). (Each angle is defined relative to an externally chosen point.) A point is in the* exterior *of the triangle if it is not in the interior and is not on any side.*

For brevity of notation, we will refer to the interiors of angles without always referencing a chosen external point. We can now define a betweenness property for rays that is exactly analogous to the betweenness property we had in Chapter 11.

**Definition 14.12.** *A ray $\overrightarrow{AD}$ is* between *rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and $D$ is interior to $\angle BAC$.*

**Theorem 14.19.** *(Crossbar Theorem) If $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then $\overrightarrow{AD}$ intersects a choice of segment $\overline{BC}$ at an interior point of $\angle BAC$.*

Proof: The proof is left as an exercise. □

The following is the converse to the preceding result.

**Theorem 14.20.** *If $D$ is a point on a segment $\overline{BC}/Z$, then there is a point $A$ not on $\overleftrightarrow{BC}$ such that $\overrightarrow{AD}$ is between $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$, for a choice of each ray.*

Proof: Axiom E-1-3 guarantees the existence of a point $A$ not on $\overleftrightarrow{BC}$. By definition, if $D$ is on $\overline{BC}/Z$, then $BC \parallel DZ$. But, this also means that $C$ is on $\overline{DZ}/B$ and $B$ is on the complimentary segment $\overline{DZ}/C$.

Thus, $D$ is interior to angle $\angle BAC$ and ray $\overrightarrow{AD}$ is between $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$. □

We note here that if we restrict our attention to only those points (and segments) that lie on a choice of segment $\overline{BC}$, then, points on that segment will obey "three-point betweenness" (proved in the exercises). Thus, by the preceding two theorems, there will be a duality between points $D$ on $\overline{BC}$ and rays $\overrightarrow{AD}$ (with $A$ not on $\overleftrightarrow{BC}$), just as we saw at the end of Project 11.2.

**Exercise 14.3.1.** *Let $A$, $B$, $C$, and $D$ be points on a line. Show that $ABC$ and $ABD$ have the same orientation if and only if $AB$ does not separate $CD$.*

**Exercise 14.3.2.** *Show that there is at least one interior point to an angle. [Hint: Use Theorem 14.3.]*

**Exercise 14.3.3.** *Prove Theorem 14.13.*

**Exercise 14.3.4.** *Using the definition of an angle, show that an angle divides the points in the Elliptic plane into three mutually exclusive groups —those that are on one of the sides of the angle, those that are interior points, and those that are exterior points. [Hint: Let $Q$ be a point that is not on a side and not interior. Use exercise 14.3.2, five-point betweenness, and prior angle theorems to show that $Q$ is exterior.].*

**Exercise 14.3.5.** *Given an angle $\angle BAC$ suppose $D$ is a point on a side of the angle (other than $A$) and $P$ is an interior point of the angle. Show that there is a choice of $\overline{DP}$ such that all points interior to that segment are interior points of the angle. [Hint: Show that $\overleftrightarrow{DP}$ intersects a side at a point $E$ and consider $\overline{DP}/E$. The proof follows similar to that of Theorem 14.16].*

**Exercise 14.3.6.** *Prove Theorem 14.19.*

## 14.4 AXIOMS OF CONGRUENCE

The axioms of congruence for Elliptic geometry are quite similar to those found in section 11.5. The definition of triangle congruence for Elliptic geometry is exactly the same as in section 11.5.

- **E-III-1 If $A$ and $B$ are distinct points on ray $\overrightarrow{AB}/X$ and $A'$ is any other point, then for each (oriented) ray $r' = \overrightarrow{A'B'}/X'$ from $A'$ there is a unique point $B'$ on $r'$ such that $B' \neq A'$ and $\overline{AB}/X \cong \overline{A'B'}/X'$.**

- **E-III-2 If $\overline{AB}/X \cong \overline{CD}/Y$ and $\overline{AB}/X \cong \overline{EF}/Z$ then $\overline{CD}/X \cong \overline{EF}/Z$. Also, every segment is congruent to itself.**

- **E-III-3 If $A * B * C/X$ on $\overline{PQ}/X$, $A' * B' * C'/X'$ on $\overline{P'Q'}/X'$, $\overline{AB}/X \cong \overline{A'B'}/X'$ , and $\overline{BC}/X \cong \overline{B'C'}/X'$, then $\overline{AC}/X \cong \overline{A'C'}/X'$. (Refer to definition 14.3 for three-point betweenness)**

- **E-III-4 If $\overline{AB}/X \cong \overline{A'B'}/X'$, then the complimentary segments are congruent. That is, $\underset{X}{\overline{AB}} \cong \underset{X'}{\overline{A'B'}}$.**

- **E-III-5 Given $\angle BAC$, defined by rays $\overrightarrow{AB}/X$ and $\overrightarrow{AC}/Y$, and given any ray $\overrightarrow{A'B'}/X'$, there are exactly two distinct rays, $\overrightarrow{A'C'}/Y'$ and $\overrightarrow{A'C''}/Y''$, such that $\angle BAC \cong \angle B'A'C'$ and $\angle BAC \cong \angle B'A'C''$.**

- **E-III-6 If $\angle BAC \cong \angle B'A'C'$ and $\angle BAC \cong \angle B''A''C''$ then $\angle B'A'C' \cong \angle B''A''C''$. Also, every angle is congruent to itself.**

- **E-III-7 Given two triangles $\Delta ABC$ and $\Delta A'B'C'$ if $\overline{AB}/X \cong \overline{A'B'}/X'$ , $\overline{AC}/Y \cong \overline{A'C'}/Y'$, and $\angle BAC \cong \angle B'A'C'$ then there is a unique choice of side $\overline{BC}$ such that the two triangles are congruent.**

The first four congruence axioms are basically the same as the axioms from section 11.5. By the work in that section, we know that segment and angle congruence will be symmetric, reflexive, and transitive operations. Axiom E-III-4 says that complimentary segments to congruent segments are themselves congruent.

Axiom E-III-5 is somewhat different than its counterpart in section 11.5. In that section, we could use plane separation to talk about the *unique* angle congruent to a given angle. In Elliptic geometry, plane separation does not hold, so the best we can do is to require that there are two possible congruent angles.

Axiom E-III-6 is also different in that we can say that two triangles are congruent in SAS congruence for a specific choice of the third side in one of the triangles.

Let's review the first few results of section 11.5 to see if they hold with these new congruence axioms. Theorem 11.27 on isosceles triangles is proved using SAS congruence on a given triangle $\Delta ABC$. The proof compares $\Delta ABC$ to $\Delta ACB$. Axiom E-III-7 can be used in just the same way for an isosceles triangle in Elliptic geometry, as these two triangles have the exact same third side, so the choice mentioned in the axiom is clear, one chooses the same side.

Supplementary angles are defined as before. The definition specifies two angles to be supplementary if they have a vertex and side in common and the other sides form a line. In Elliptic geometry the rays forming a side are also lines, so supplementary angles must be defined on the same sides. This implies that the supplementary angle to an angle $\angle BAC$, defined in relation to an exterior point $Z$, must be the dual choice of angle defined in the discussion following definition 14.9.

A quick review of the proof of Theorem 11.28 shows that the proof relies on the existence of two oppositely oriented rays from a point and three-point betweenness. As seen earlier in this chapter, there is a way to define opposite rays in Elliptic geometry, and by Theorem 14.10 we know that we can always embed a segment (or three points) inside a larger segment for which three-point betweenness holds. Thus, with some adjustment, the proof of Theorem 11.28 is valid in Elliptic geometry.

Vertical angles are defined as before and the Elliptic version of Theorem 11.29 follows immediately from Theorem 11.28.

The proof of ASA congruence (Theorem 11.30) will hold in Elliptic geometry, with the proviso that the conclusion of the theorem statement must be adjusted to say that the two triangles are congruent with a *choice* of the remaining sides of one of the triangles. The Elliptic version of Theorem 11.31 follows directly from Theorem 11.30.

The results on segment ordering in section 11.5.2 assume "three-point" betweenness. A close reading of the proofs in that section shows that they require only the notions of three-point betweenness, the notion of opposite rays from a point, and the notion of interior points to triangles. By suitable restriction to a particular Elliptic segment, we can assume three-point betweenness holds on that segment, and we have already provided suitable substitute definitions for opposite rays and interior points to angles. Thus, the results from section 11.5.2 hold in Elliptic geometry.

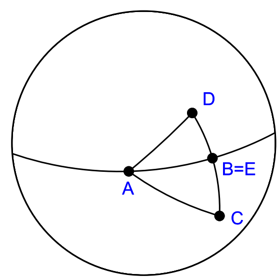
Similarly, all of the results from Project 11.5.2, up to Theorem 11.40, will hold in Elliptic geometry. These results include properties of angle order, addition, and subtraction. Theorem 11.40 has the assumption that "two points $C$ and $D$ are on opposite sides of a line $\overleftrightarrow{AB}$," whereas Theorem 11.41 has the assumption that $C$ and $D$ are on the same side of $\overleftrightarrow{AB}$. Since plane separation is not possible in Elliptic geometry, these theorems cannot be directly translated to Elliptic geometry. However, the results of both theorems can be simplified to one single statement as follows:

**Theorem 14.21.** *If two points $C$ and $D$ are not on line $\overleftrightarrow{AB}$ and if $\overline{AC} \cong \overline{AD}$ and $\overline{BC} \cong \overline{BD}$ then $\angle ABC \cong \angle ABD$ and $\angle BAC \cong \angle BAD$, and $\Delta ABC \cong \Delta ABD$.*

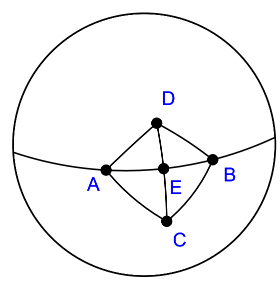
Proof: If $C = D$, the the result is clear. If $C \neq D$, then the proof will follow similarly to the proof of Theorem 11.40.

Since $C$ and $D$ are not collinear with $\overleftrightarrow{AB}$, then one choice of $\overline{CD}$ will intersect $\overleftrightarrow{AB}$ at some point $E$.

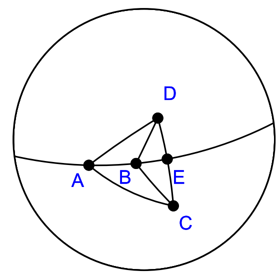
If $E = B$, then $\Delta ADC$ is an isosceles triangle. Thus, $\angle ACB \cong \angle ADB$. By SAS we have that $\Delta ABC \cong \Delta ABD$. The case where $E = A$ can be handled similarly.




If $E$ does not coincide with $A$ or $B$, then either $\angle ADB > \angle AEB$ (shown here) or $\angle ADB < \angle AEB$. Suppose $\angle ADB > \angle AEB$. Then, $\Delta ADC$ and $\Delta BDC$ are isosceles and so $\angle ADE \cong \angle ACE$ and $\angle BDE \cong \angle BCE$. By the angle sum property, $\angle ADB \cong \angle ACB$. Then, by SAS we have $\Delta ABC \cong \Delta ABD$.



If $\angle ADB < \angle AEB$, we again use isosceles triangles, and the angle subtraction property, to show $\Delta ABC \cong \Delta ABD$.




□

The Elliptic version of Side-Side-Side Triangle Congruence can be proven using this result. The proof is left as an exercise.

**Exercise 14.4.1.** *Prove SSS congruence in Elliptic geometry. [Hint: Refer to the proof outlined in exercise 11.6.5.]*

**Exercise 14.4.2.** *Prove that Theorem 11.48 holds in Elliptic geometry. [Hint: The proof can follow that of Theorem 11.48, using Axiom E-III-5 where appropriate].*

**Exercise 14.4.3.** *Modify the proof of Theorem 11.28 so that it is valid in Elliptic Geometry.*

**Exercise 14.4.4.** *Provide a counter-example in the sphere model of Elliptic Geometry to show that AAS is not valid in Elliptic Geometry.*

## 14.5 CONSTRUCTIONS AND DEDEKIND'S AXIOM

We now consider the construction results from sections 11.7 and the segment and angle measure results from sections 11.8 and 11.9.

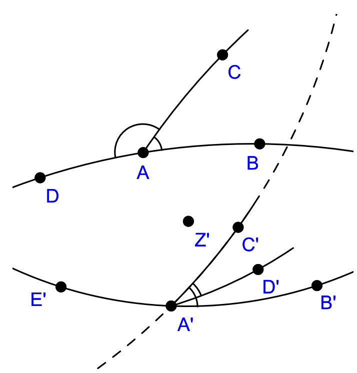
Theorem 11.43 states that an isosceles triangle can always be constructed on a given segment. The proof of this theorem will hold in Elliptic geometry, as it is based on incidence properties, angle order, the Crossbar Theorem, and basic results on isosceles triangles. All of these properties have been shown to be true in Elliptic geometry.

Theorem 11.44 relies on the idea of plane separation, and thus cannot be directly translated to Elliptic geometry. However, a workable substitute is the following:

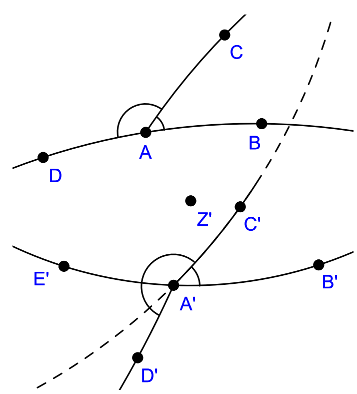
**Theorem 14.22.** *Suppose we have two supplementary angles $\angle BAC$ and $\angle CAD$ on line $\overleftrightarrow{BD}$. Also, suppose that $\angle BAC \cong \angle B'A'C'$ and $\angle CAD \cong \angle C'A'D'$. Suppose that angles $\angle B'A'C'$ and $C'A'D'$ are not the same angle. Then, angles $\angle B'A'C'$ and $\angle C'A'D'$ are supplementary.*

Proof: By exercise 14.3.4 we know that $D'$ is either on one of the sides of $\angle B'A'C'$, or it is an interior point of this angle, or it is an exterior point.

Suppose $D'$ is interior to $\angle B'A'C'$. By the theorems on angle order, since $D'$ is interior to $\angle B'A'C'$, then $\angle C'A'D' < \angle B'A'C'$. Since $\angle BAC \cong \angle B'A'C'$ and $\angle CAD \cong \angle C'A'D'$, then $\angle CAD < \angle BAC$. But, this means that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This contradicts the fact that $D$ is on $\overrightarrow{AD}$.




Suppose $D'$ is exterior to $\angle B'A'C'$. Then, by the duality of angles (discussed when we defined angles), we know that $D'$ will be an interior point to the dual choice of $\angle B'A'C'$, say $\angle C'A'E'$. In the figure at the right we have illustrated this possibility in our circle model of Elliptic geometry.




By the theorems on angle order, since $D'$ is interior to $\angle C'A'E'$, then $\angle C'A'D' < \angle C'A'E'$. But, by the Elliptic geometry version of Theorem 11.28, we know that supplementary angles of congruent angles are congruent and so $\angle C'A'E' \cong \angle CAD$. By angle ordering, this implies that $\angle C'A'D' < \angle CAD$. We are given that $\angle CAD \cong \angle C'A'D'$. Our results on angle ordering say that only one of $\angle C'A'D' < \angle CAD$,

$\angle C'A'D' > \angle CAD$, or $\angle CAD \cong \angle C'A'D'$ can hold. This contradiction shows $D'$ cannot be exterior to $\angle B'A'C'$.

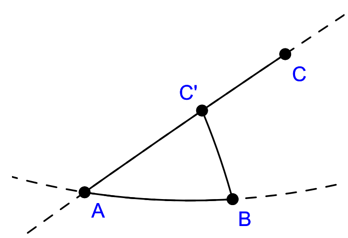
The only possibility left for $D'$ is that it is on one of the sides of $\angle B'A'C'$. It cannot be on $\overrightarrow{A'C'}$ by the definition of $\angle C'A'D'$. Thus, $D'$ is on $\overrightarrow{A'B'}$. Since angles $\angle B'A'C'$ and $C'A'D'$ are not the same angle, then $\angle C'A'D'$ must be supplementary to $\angle B'A'C'$. □

This theorem can be used to prove that any angle can be bisected.

**Theorem 14.23.** *Given angle $\angle BAC$ we can find a ray $\overrightarrow{AD}$ between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $\angle BAD \cong \angle DAC$.*

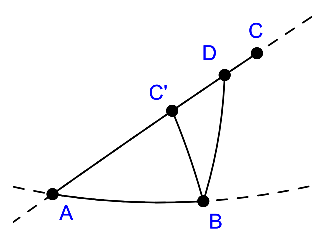
Proof:

By using properties of extension of segments, we can assume that $\overline{AC} > \overline{AB}$. On $\overline{AC}$ we can find $C'$ such that $\overline{AB} \cong \overline{AC'}$ ( Axiom E-III-1).



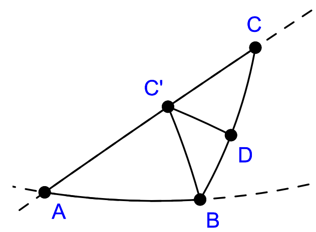

Following the proof of Theorem 11.43, we can construct an isosceles triangle $\Delta BC'D$ on $\overline{BC'}$, using point $C$. (We assume that the choice of $\overline{BC'}$ is such that the interior points on $\overline{BC'}$ are interior to $\angle BAC$. See Theorem 14.16) From the construction, we have that $\overline{C'D} \cong \overline{BD}$ and $D$ is either on $\overline{CC'}$ or $\overline{BC}$.

Suppose that $D$ is on $\overline{CC'}$. Then, $\angle BC'D$ is supplementary to $\angle AC'B$. Since $\Delta BC'D$ is isosceles, then $\angle BC'D \cong \angle DBC'$. Also, since $\Delta ABC'$ is isosceles, then $\angle C'BA \cong \angle AC'B$.




So, we have that $\angle BC'D$ and $\angle AC'B$ are supplementary, $\angle BC'D \cong \angle DBC'$, and $\angle AC'B \cong \angle C'BA$. Also, since $D$ is not on $\overleftrightarrow{AB}$ we have that $\angle C'BA$ is not the same angle as $\angle DBC'$. Thus, by the previous theorem, we have that $\angle C'BA$ and $\angle DBC'$ are supplementary. But, this implies that $D$ is on $\overleftrightarrow{AB}$ which contradicts incidence axiom E-I-2.

Since $D$ cannot be on $\overline{CC'}$, then it must be on $\overline{BC}$, and so $\overrightarrow{AD}$ is interior to $\angle BAC$. Using SSS congruence on triangles $\Delta DBA$ and $\Delta DC'A$ we have that $\angle BAD \cong \angle DAC'$. □




Theorem 11.46 ( the construction of a segment bisector) will now follow directly, as it's proof relies on the construction of isosceles triangles, angle bisectors, and the Crossbar Theorem.

Perpendicular lines are defined the same way as before. Theorems 11.47 and 11.48 deal with the construction of perpendiculars to a line through points on the line and points off the line. The proof of Theorem 11.47 transfers directly to Elliptic geometry. The proof of Theorem 11.48 requires a bit more work, but the proof only needs a slight adjustment and will be left as an exercise.

### Dedekind's Axiom

The construction for segment and angle measure found in sections 11.8 and 11.9 fundamentally depended on Dedekind's axiom for (unbounded) lines. Since lines in Elliptic geometry are bounded, we need to adjust this axiom so that it applies to segments. With this restriction, we can use the three-point betweenness language as we did in Chapter 11 with no problems.

- **IV-1 (Dedekind's Elliptic Axiom) If the points on a segment $s$ are partitioned into two nonempty subsets $\Sigma_1$ and $\Sigma_2$ (i.e. $l = \Sigma_1 \cup \Sigma_2$) such that no point of $\Sigma_1$ is between two points of $\Sigma_2$ and vice-versa, then there is a unique point $O$ lying on $l$ such that $P_1 * O * P_2$ if and only if one of $P_1$ or $P_2$ is in $\Sigma_1$, the other is in $\Sigma_2$, and $O \neq P_1$ or $P_2$.**

Dedekind's axiom says that any splitting of a segment into points that are on distinct opposite sides must be accomplished by a unique point $O$ acting as the separator. The pair of subsets described in the axiom is called a *Dedekind cut* of the segment.

With this new version of Dedekind's axiom, all of the results in section 11.8 can be shown to be valid, with some slight modifications of definitions.

For example, let's consider the notion of "laying off" a segment on a ray. In Elliptic geometry, we will replace this idea with the following:

**Definition 14.13.** *We say that segment* $\overline{CD}$ *is* laid off $n$ times *($n$ a positive integer) on a segment* $\overline{AB}$ *if there is a sequence of points* $A_0 = A, A_1, A_2, \ldots, A_n$ *on* $\overline{AB}$ *with* $\overline{A_{k-1}A_k} \cong \overline{CD}$ *for* $k = 1 \ldots n$ *and* $A * A_k * A_{k+1}$ *for* $k = 1 \ldots n-1$*. We also write* $n\overline{CD}$ *for laying off* $\overline{CD}$ $n$ *times.*

Note that this definition makes sense, as three-point betweenness has its usual properties if we restrict all constructions to a single segment $\overline{AB}$.

The following lemma is analogous to Lemma 11.49 from section 11.8 with a slight change from rays to segments.

**Lemma 14.24.** *Let segment* $\overline{CD}$ *be laid off* $n$ *times on* $\overline{AB}$*. Let* $\{A_k\}_{k=0}^n$ *be the corresponding sequence of points on* $\overline{AB}$*. Then,* $A * A_j * A_k$ *for all* $j = 1 \ldots n-1$*,* $k = 2 \ldots n$*, with* $j < k$*.*

Proof: Exercise. □

We can now state and prove Archimedes' Axiom, in almost the same fashion as we did in Chapter 11. In fact, we show in bold the parts of the proof that are changed.

**Theorem 14.25.** *(Archimedes's axiom) Given* $\overline{AB}$ *and* $\overline{CD}$*, there is a positive integer* $n$ *such that if we lay off* $\overline{CD}$ $n$ *times on* $\overline{AB}$*, starting from* $A$*, then a point* $A_n$ *is reached where either* $A_n = B$ *or* $A_n$ *is exterior to* $\overline{AB}$*.*

Proof: Suppose that no such $n$ exists, i.e. for all $n > 1$ the point $A_n$ reached by laying off $\overline{CD}$ $n$ times is **always interior to** $\overline{\mathbf{AB}}$. Then, for all $n$ we have $A * A_n * B$. We will define a Dedekind cut for $\overline{\mathbf{AB}}$ as follows.

Let $\Sigma_1$ be the set of points $P$ on $\overline{\mathbf{AB}}$ such that $A * A_n * P$ for all $n$. Then, $B$ is in $\Sigma_1$ and $\Sigma_1$ is non-empty. Let $\Sigma_2$ be the set of remaining points on $\overline{\mathbf{AB}}$ . For all $n$, $A_n$ is in $\Sigma_2$ and $\Sigma_2$ is non-empty. Also note that $A$ is in $\Sigma_2$.

We now show that the betweenness condition in Dedekind's axiom is satisfied. Let $Q_1, R_1$ be two points of $\Sigma_1$ and $Q_2, R_2$ be points of $\Sigma_2$. Suppose that $Q_2 * Q_1 * R_2$. Since $Q_2$ and $R_2$ are not in $\Sigma_1$, then for some $n_1$, we must have $A * Q_2 * A_{n_1}$, and for some $n_2$, we have $A * R_2 * A_{n_2}$. If $n_1 = n_2$ we can use the fact that $A * A_{n_2} * A_{n_2+1}$, and four-point betweenness, to show that $A * R_2 * A_{n_2+1}$. Thus, we can assume that $n_1 \neq n_2$, and without loss of generality, that $n_2 > n_1$. By the previous lemma we know that $A * A_{n_1} * A_{n_2}$. Since $A * Q_2 * A_{n_1}$, then $A * Q_2 * A_{n_2}$ by 4-point betweenness. Using $A * Q_2 * A_{n_2}$, $A * R_2 * A_{n_2}$, and $A * A_{n_2} * Q_1$ we have by 4-point betweenness that $A * Q_2 * Q_1$ and $A * R_2 * Q_1$. Thus, $Q_2$ and $R_2$ are on the same side of $Q_1$. But, this contradicts the assumption that $Q_2 * Q_1 * R_2$, and so a point of $\Sigma_1$ cannot be between two points of $\Sigma_2$.

Suppose on the other hand that $Q_1 * Q_2 * R_1$. A similar argument to the previous one will show that $Q_1$ and $R_1$ are on the same side of $Q_2$, which is again a contradiction.

Thus, the conditions for Dedekind's **Elliptic** Axiom are satisfied and there must be a unique point $O$ with the properties stated in the axiom. If $O = A_n$ for some $n$, then $A * O * A_{n+1}$ would imply by the axiom that $A_{n+1}$ is in $\Sigma_1$, which is impossible. If $O \neq A_n$, but $A * O * A_k$ for some $k$ then $O$ would be between two points of $\Sigma_2$, which would also contradict Dedekind's axiom. Thus, $O$ must be in $\Sigma_1$.

Now, $O$ is on the same side of $A$ as $A_n$ (for any $n$), for if $O$ was on the other side for some $n$, then $O * A * A_n$, which contradicts the fact that $O$ is in $\Sigma_1$. Also, $\overline{AO} > \overline{CD}$, for if $\overline{AO} < \overline{CD}$, then $A * O * A_1$, and $O$ would be in $\Sigma_2$.

Now, we will show that the existence of point $O$ leads to a contradiction. First, there is a point $X$ with $A * X * O$ and $\overline{XO} \cong \overline{CD}$, (Congruence axiom **E-III-1**). Also, $X \neq A_n$ for any $n$ since, if it did match one of the $A_n$, then $O = A_{n+1}$, and $O$ would be in $\Sigma_2$, which is a contradiction. For $P$ in $\Sigma_1$, we have $A * O * P$. Since $A * X * O$, then by 4-point betweenness we have $X * O * P$. By Dedekind's **Elliptic** Axiom $X$ must be in $\Sigma_2$. Thus, there is an $n > 0$ such that $A * X * A_n$. Since $A * A_n * O$ we have by 4-point betweenness that $X * A_n * O$. By the previous lemma $A * A_n * A_{n+2}$. Thus, by 4-point betweenness we have $A * X * A_{n+2}$. Since $A * A_{n+2} * O$ we have again by 4-point betweenness that $X * A_{n+2} * O$. By segment ordering we have $\overline{XO} > \overline{XA_{n+2}}$. Now, since $A * X * A_n$ and $A * A_n * A_{n+2}$ then by 4-point betweenness we have $X * A_n * A_{n+2}$. Thus, $\overline{XA_{n+2}} > \overline{A_nA_{n+2}}$. By transitivity of segment ordering we have $\overline{XO} > \overline{A_nA_{n+2}}$. But, $\overline{A_nA_{n+2}} > \overline{A_nA_{n+1}}$ and $\overline{A_nA_{n+1}} \cong \overline{CD} \cong \overline{XO}$.

Thus, $\overline{XO} > \overline{XO}$. Since a segment cannot be larger than itself we have a contradiction. This completes the proof. □

We now state the corresponding theorems for Elliptic geometry from the rest of section 11.8. The proofs of these theorems require only slight changes, like we did in the proof of Archimedes' Axiom, and so will be omitted or carried out in the exercises.

**Theorem 14.26.** *Let $\overline{A_nB_n}$ be a nested sequence on a given segment $\overline{AB}$. Then $\overline{A_nB_n} \subset \overline{A_mB_m}$ for all $n > m$. Also, $A_m * A_r * B_n$ and $A_n * B_r * B_m$ for any $n, m$ with $r > m$.*

**Theorem 14.27.** *(Cantor's Axiom) Suppose that there is an infinite nested sequence of segments $\overline{A_nB_n}$ $(n > 0)$ on a segment $\overline{AB}$. Suppose there does not exist a segment which is less than all of the segments $\overline{A_nB_n}$. Then, there exists a unique point $O$ belonging to all the segments $\overline{A_nB_n}$.*

The notion of "adding" segments needs no modification, as the sum is restricted to a given segment.

**Definition 14.14.** *Given segments $a = \overline{AA'}$, $b = \overline{BB'}$, and $c = \overline{CC'}$, we say that $c$ is the* sum *of $a$ and $b$, denoted $c = a + b$, if there exists a point $X$ with $C * X * C'$, $\overline{AA'} \cong \overline{CX}$, and $\overline{BB'} \cong \overline{XC'}$. If we refer to $a + b$, then it is implicitly assumed that there exists a segment $c$ such that $c = a + b$.*

**Theorem 14.28.** *Given segments $a, b, c, d$ we have*

*(i)* $a + b = b + a$

*(ii)* $(a + b) + c = a + (b + c)$

*(iii)* *if* $a < b$ *then* $a + c < b + c$

*(iv)* *if* $a < b$ *and* $c < d$ *then* $a + c < b + d$

*(v)* *if* $a = b$ *and* $c = d$ *then* $a + c = b + d$

Dyadic numbers are defined exactly the same as in Chapter 11, with the proviso that for a given segment $a$, the dyadic number $\frac{m}{2^n}a$ is not necessarily defined for all $m$. With this understanding, we have

**Theorem 14.29.** *Let $w$ and $v$ be dyadic numbers and $a$ and $b$ segments. Then,*

*(i)* $wa = wb$ *iff* $a = b$.

*(ii)* $w(a + b) = wa + wb$

*(iii)* $(w + v)a = wa + va$

*(iv)* *if* $a < b$ *then* $wa < wb$

*(v)* *if* $w < v$ *then* $wa < va$

*(vi)* *if* $wa < wb$ *then* $a < b$

*(vii)* *if* $wa < va$ *then* $w < v$

Segment measure can be defined in Elliptic geometry, just as it was in Chapter 11.

**Theorem 14.30.** *Given a segment $u$, which we will call a* unit segment*, there is a unique way of assigning a positive real number, called the* length *and denoted by $\mu(a)$, to any segment $a$, such that for all segments $a$ and $b$ we have*

*(i)* $\mu(a) > 0$ *for all* $a$.

*(ii)* $a \cong b$ *iff* $\mu(a) = \mu(b)$.

*(iii)* $a < b$ *iff* $\mu(a) < \mu(b)$.

*(iv)* $\mu(a + b) = \mu(a) + \mu(b)$.

*(v)* $\mu(u) = 1$.

The major difference with segment measure in Elliptic geometry, is that there is no guarantee that it is an unbounded measure. In fact, in Chapter 8, we showed that once we had a segment measure function defined, then we could prove that all lines have finite length.

For angle measure, we make note of the fact that the betweenness properties for rays can be put into a one-to-one correspondence with the betweenness properties for segments. Thus, all of the material on angle measure in section 11.9 will hold in Elliptic geometry, with small adjustments to proofs as we have shown above. We will not recall all of these results here, but refer the reader to the end of section 11.9.

### Line and Circle Continuity

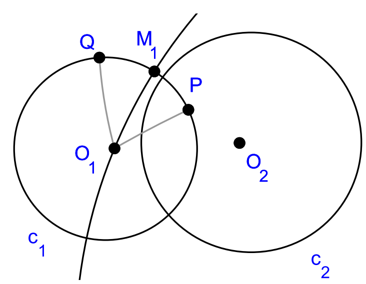
Circles are defined in Elliptic geometry the same way they were defined in definition 11.28.

Circle-Circle Continuity holds in Elliptic geometry.

**Theorem 14.31.** *(Circle-Circle Continuity) Given two circles $c_1$ and $c_2$, with centers $O_1$ and $O_2$, if $c_1$ contains a point $P$ inside of $c_2$ and also contains a point $Q$ outside of $c_2$, then there are exactly two distinct points of $c_1$ that are also on $c_2$.*

Proof:

Let $O_1$ be the center of circle $c_1$ and let $O_2$ be the center of circle $c_2$. Consider $\angle PO_1Q$. Let $\overrightarrow{O_1M_1}$ be the angle bisector. We can assume $M_1$ is on $c_1$. If $\overline{O_2M_1}$ has measure equal to the radius of $c_2$ we have found one intersection. Otherwise, either the measure of $\overline{O_2M_1}$ is greater than the radius of $c_2$ or less than the radius of $c_2$.




If the measure of $\overline{O_2M_1}$ is greater than the radius of $c_2$, than we will consider $\angle PO_1M_1$, where $P$ is inside $c_2$ and $M_1$ is outside $c_2$. If the measure of $\overline{O_2M_1}$ is less than the radius of $c_2$, than we will consider $\angle M_1O_1Q$, where $M_1$ is inside $c_2$ and $Q$ is outside $c_2$. In either case, we now consider an angle whose measure is half that of $\angle PO_1Q$.

We now find the angle bisector $\overrightarrow{O_1M_2}$ (with $M_2$ on $c_1$) of this new angle and again test to see if $M_2$ is on $c_2$. If so, we have found one intersection. If not, we again look at one of the two half-angles with a point inside $c_2$ and a point outside $c_2$. We continue sub-dividing in this fashion, yielding a sequence of angles whose measures have length $\frac{1}{2^m}$ times the measure of $\angle PO_1Q$. By the continuity of the reals, and of angle measure, this sequence has angles whose measures approach zero, and thus there must be a ray $\overrightarrow{O_1I}$ (with $I$ on $c_1$) common to all the sequence terms. We claim that the length of $\overline{O_2I}$ is equal to the radius of $c_2$. To prove this claim we will use the distance function from a point to an elliptic segment.

The distance function from a point to an elliptic segment can be shown to be a continuous function. The proof uses the trigonometry of elliptic right triangles. We will not take the time here to carry out this development, but a full explanation can be found in [19] Chapter 7. The proof only depends on results we have already shown for elliptic angles and segments and also depends on the continuity of segment and angle measure, which has been shown earlier in this chapter.

Assuming the continuity of the distance function, consider each angle in the bisection process discussed above. Let this angle be $P_mO_1Q_M$, where $P_M$ and $Q_M$ are on $c_1$ and $P_m$ is inside $c_2$ and $Q_m$ is outside $c_2$. Let $r_2$ be the radius of $c_2$. Then, the distance from $O_2$ to $P_m$ is less than $r_2$ and the distance from $O_2$ to $Q_m$ is greater than $r_2$. By the continuity of the distance from $O_2$ to the segment $\overline{P_mQ_m}$ there must be a point $I_m$ on this segment such that the distance from $O_2$ to $I_m$ equals $r_2$.

Now, as the sequence of angles approaches zero, the segments $\overline{P_mQ_m}$ must have measures approaching zero. Thus, these segments must approach a point $I$ and all the $I_m$'s must also approach $I$. □

Just as before, we can use this axiom to prove Euclid's Proposition 1 on the construction of equilateral triangles.

We conclude that Euclid's Propositions 1-15, 23, and the ASA triangle congruence result all hold in Elliptic geometry. It would be natural to ask whether the same is true for Euclid's Proposition 16 – the Exterior Angle Theorem. In Chapter 8 we saw that in Elliptic geometry it is possible to construct a triangle with two right angles. For such a triangle, the Exterior Angle Theorem fails. The material in Chapter 8 was carefully laid out so that all arguments and proofs were based solely on Euclid's Propositions 1-15, 23, and the ASA triangle congruence. We also made use of results about reflections and rotations from Chapter 5, as these were based solely on Euclid 1-15, 23, and ASA.

**Exercise 14.5.1.** *Prove Lemma 14.24.*

**Exercise 14.5.2.** *Find the places in the proof of Theorem 11.52 from Chapter 11 that need modification to prove Theorem 14.27. List these changes.*

**Exercise 14.5.3.** *Find the places in the proof of Theorem 11.53 from Chapter 11 that need modification to prove Theorem 14.28. List these changes.*

**Exercise 14.5.4.** *Find the places in the proof of Theorem 11.54 from Chapter 11 that need modification to prove Theorem 14.29. List these changes.*

**Exercise 14.5.5.** *Find the places in the proof of Theorem 11.55 from Chapter 11 that need modification to prove Theorem 14.30. List these changes.*