CHAPTER 13

Foundations of Euclidean Geometry

> I've always been passionate about geometry.
>
> – Erno Rubik (Rubik's Cube Inventor) (1944–)

13.1 PARALLELS

In Chapters 11 and 12, we have developed an axiomatic foundation for Universal and Neutral geometry. We have shown that this axiomatic foundation can be used to prove the first 28 of Euclid's propositions in Book I of *Elements*, along with Proposition 31. A sound development of the remaining propositions (29, 30, 32–48) can now be carried out, if one adds a final axiom, the parallel axiom. In Euclid's original axiomatic system, the parallel axiom is rather wordy and not very intuitive:

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

We will replace this axiom with a logically equivalent axiom —Playfair's axiom. (Refer to section 2.1 for a proof of this equivalence).

- **Playfair's Axiom** Given a line l and a point P not on l, it is possible to construct exactly one line through P parallel to l.

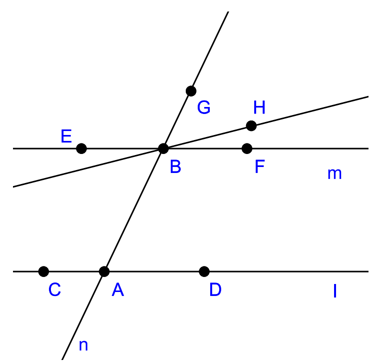
Playfair's axiom can be used to prove Euclid's Proposition 29.

Theorem 13.1. *(Proposition 29) If a line n intersects two parallel lines l and m, then alternate interior angles are congruent, corresponding angles are congruent, and the interior angles on the same side of n are supplementary.*

Proof: Let n intersect l at A and m at B. Let C, D be points on opposite sides of A on l and let E, F be points on opposite sides of B on m. Assume that D, F are on the same side of n, and C, E are on the same side of n.

We start by proving the result concerning corresponding angles. Let $\angle FBG$ be the corresponding angle to $\angle DAB$.

Assume $\angle FBG$ is not congruent to $\angle DAB$. We may assume that $\angle FBG$ is greater than $\angle DAB$. By congruence axiom III-4 we can create $\angle HBG$ on $\overrightarrow{BG}$ so that $\angle HBG \cong \angle DAB$. By Theorem 12.12 $\overleftrightarrow{BH}$ is parallel to $\overleftrightarrow{AD}$. But this means there are two different lines through B that are parallel to $\overleftrightarrow{AD}$, which contradicts Playfair's Postulate.



Now, consider alternate interior angles $\angle DAB$ and $\angle EBA$. We have already shown that $\angle DAB \cong \angle FBG$. By the Vertical Angles Theorem (Theorem 11.29) we know that $\angle FBG \cong \angle EBA$. Thus, $\angle DAB \cong \angle EBA$.

The final part of the proof concerning interior angles on the same side of n is left as an exercise. □

Proposition 30 continues the exploration of parallelism. The proof of this result will be left as an exercise.

Theorem 13.2. *(Proposition 30) Two lines that are each parallel to a third line must be the same line or are parallel to one another.*

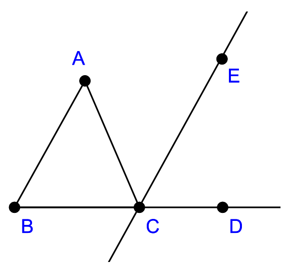
Proposition 31 states that one can construct a parallel to a given line through a point not on that line. This was proven in the previous section, as this result is part of Neutral geometry.

Euclid uses Proposition 31 to prove that the sum of the angles in a triangle is equal to two right angles (Proposition 32). The following theorem can substitute for Proposition 32, as it is logically equivalent to that proposition.

Theorem 13.3. *Given* ΔABC. *Let* $\overline{BC}$ *be extended to point* D *so that* $\angle DCA$ *is an exterior angle to the triangle. Then,* $\angle DCA$ *is congruent to the addition of* $\angle CBA$ *and* $\angle BAC$.

Proof:

By Corollary 12.16 (Euclid's Proposition 31), we can construct $\overleftrightarrow{CE}$ through E parallel to $\overleftrightarrow{AB}$. Since $\overleftrightarrow{AC}$ crosses $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$, then by Theorem 13.1 we have that $\angle BAC \cong \angle ECA$. Also, since $\overleftrightarrow{BD}$ crosses $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$, then by Theorem 13.1 we have that $\angle DCE \cong \angle CBA$.



Now, the exterior angle $\angle DCA$ is the sum of $\angle DCE$ and $\angle ECA$ (definition of angle addition). Also, $\angle DCE \cong \angle CBA$ and $\angle ECA \cong \angle BAC$. Thus, by Theorem 11.36 we have that $\angle DCA$ is congruent to the sum of angles $\angle CBA$ and $\angle BAC$. □

13.1.1 Parallelograms

Euclid's Propositions 33 and 34 deal with parallelograms in the plane. To define a parallelogram, we first need to review the definition of a quadrilateral.

In definition 2.18 we defined a quadrilateral $ABCD$ as a figure comprised of segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ such that no three of the points of the quadrilateral are collinear and no pair of segments intersects, except at the endpoints.

A parallelogram is then a quadrilateral where the lines defined by opposite sides ($\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, or $\overleftrightarrow{BC}$ and $\overleftrightarrow{DA}$) are parallel.

Euclid's original wording of Proposition 33 is as follows:

> The straight lines joining equal and parallel straight lines (at

> the extremities that are) in the same directions (respectively) are themselves equal and parallel.

The parallel "straight lines" in this statement are two segments, say $\overline{AB}$ and $\overline{CD}$, that lie on two parallel lines. We will interpret "equal" to mean that these segments are congruent. To construct two segments "joining" the given segments at the "extremities" will be to construct the segments $\overline{BC}$ and $\overline{AD}$. As shown in Figure 13.1, there is some ambiguity as to how these new segments can be constructed.

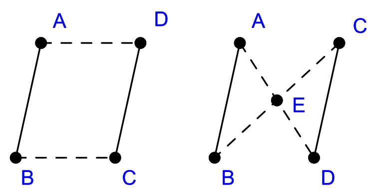


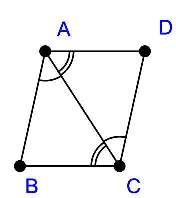
Figure 13.1

There are two possibilities. Either the segments $\overline{BC}$ and $\overline{AD}$ intersect at a point E, which is necessarily between B and C (and also between A and D), or these two segments do not intersect. Euclid uses the phrase "in the same direction" to try and specify the configuration of the points, but this phrase really has no meaning. A simple fix is to just assume that the four-point figure $ABCD$ is a quadrilateral. The following theorem is a bit stronger result than Euclid's Proposition 33.

Theorem 13.4. *Quadrilateral $ABCD$ is a parallelogram if and only if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and $\overline{AB} \cong \overline{CD}$.*

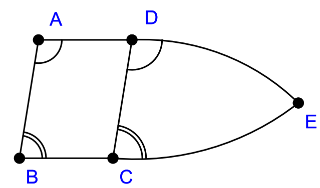
Proof: Assume that $ABCD$ is a parallelogram. Then, by the definition of parallelograms, we have that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are parallel.

Construct $\overline{AC}$. Then, by using Theorem 13.1 on each pair of parallel lines, we have $\angle BAC \cong \angle ADC$ and $\angle ACB \cong \angle DAC$. By ASA congruence we have that $\Delta BAC \cong \Delta DCA$ and thus $\overline{AB} \cong \overline{CD}$.



Now, suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and $\overline{AB} \cong \overline{CD}$. Suppose that $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are not parallel. Then, they intersect at a point E.

By Theorem 13.1 we have $\angle BAD \cong \angle CDE$ and $\angle CBA \cong \angle ECD$. By ASA congruence, $\Delta BAE \cong \Delta CDE$ and thus $\overline{BE} \cong \overline{CE}$.



But, by the discussion prior to this theorem we know that E is not between B and C (and also not between A and D). By segment ordering, $\overline{BE} \ncong \overline{CE}$, So, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ must be parallel. □

Proposition 34 also relates to parallelograms.

Theorem 13.5. *(Proposition 34) If $ABCD$ is a parallelogram, then both pairs of opposite sides are congruent, and both pairs of opposite angles are congruent. Also, the diagonals split the parallelogram into two congruent triangles.*

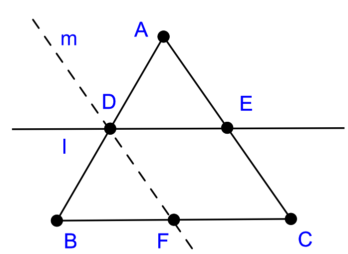
Proof: Opposite sides and diagonals are defined in definition 2.18 and definition!2.19. The proof will be carried out in the exercises. □

Before we move on to Proposition 35 and the notion of area in Euclidean geometry, we will look at a couple of elegant results that make use of our work on parallelograms.

Theorem 13.6. *Let ΔABC be a triangle and let D be the midpoint of $\overline{BC}$. Then, the parallel to $\overleftrightarrow{BC}$ through D intersects $\overline{AC}$ at the midpoint of that side.*

Proof:

We know the midpoint D of $\overline{AB}$ exists by Theorem 11.46. Let l be the parallel to $\overleftrightarrow{BC}$ through D. By Pasch's axiom, since l does not intersect $\overline{BC}$, then it must intersect $\overline{AC}$ at a point E.



Let m be the parallel to $\overleftrightarrow{AC}$ through D. By Pasch's axiom, this intersects $\overline{BC}$ at a point F. Since corresponding angles of parallel lines are congruent, we have $\angle ADE \cong \angle DBF$. Likewise, $\angle DAE \cong \angle BDF$. By ASA, $\Delta ADE \cong DBF$, and so $\overline{AE} \cong \overline{DF}$.

Consider parallelogram $DECF$. We have proven that opposite sides are congruent. Thus, $\overline{DF} \cong \overline{CE}$. Since $\overline{AE} \cong \overline{DF}$, then $\overline{AE} \cong \overline{CE}$ and E is the midpoint of $\overline{AC}$. □

Theorem 13.7. *Let l, m, and n be three parallel lines and let t be a transversal to these lines that crosses l at A, m at B, and n at C. Suppose that $\overline{AB} \cong \overline{BC}$. Then, any other transversal crosses these lines at points A', B', and C' where $\overline{A'B'} \cong \overline{B'C'}$ (Figure 13.2).*

Proof: The proof is left as an exercise. □

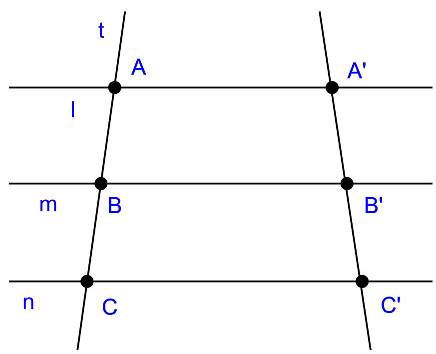


Figure 13.2

Exercise 13.1.1. *Finish the proof of Theorem 13.1. [Hint: It is enough to show that angles $\angle DAB$ and $\angle ABF$ are supplementary.]*

Exercise 13.1.2. *Prove Theorem 13.2. [Hint: Try a proof by contradiction.]*

Exercise 13.1.3. *In Euclid's original wording of Proposition 32, he states:*

> *In any triangle, if one of the sides be produced, the exterior angle is equal to the two opposite and interior angles, and the three interior angles of the triangle are equal to two right angles.*

Assume there is the usual way of associating an angle with a real number, which we will call the degree measure *of the angle. Use Theorem 13.3 to prove that the sum of the angles in a triangle adds to 180 degrees.*

Exercise 13.1.4. *Show that one can prove the result stated in Euclid's original version of Proposition 33 by using Theorem 13.4. (You will need to make the same assumptions we made in the discussion about Euclid's wording of Proposition 33)*

Exercise 13.1.5. *Theorem 13.5.*

Exercise 13.1.6. *Prove Theorem 13.7. [Hint: Use Theorem 13.6 twice.]*

Exercise 13.1.7. *Let $ABCD$ be a quadrilateral. Show that the figure realized by joining the midpoints of the four sides is a parallelogram. [Hint: Use Theorem 13.6.]*

13.2 SEGMENT MULTIPLICATION AND SIMILARITY

In section 11.8 we carefully defined segment addition by geometric construction. We then showed that we could define segment measure so that segment addition was equivalent with real number addition. We will take a break from our survey of Euclid's propositions to define segment *multiplication* in Euclidean geometry.

To start with, we assume we have a well-defined segment measure, with a special unit segment u having length 1. To define segment multiplication, we will need the following constructions:

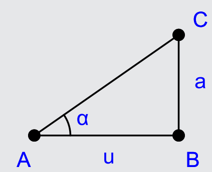
Lemma 13.8. *Let a and b be two segments. Then, there is a right triangle ΔABC, with right angle at B, having legs (sides opposite the non-right angles) congruent to a and b.*

Proof: Exercise. □

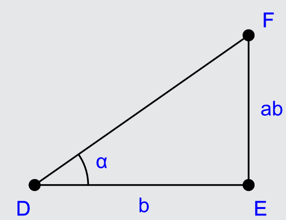
Lemma 13.9. *Let a be a leg in a right triangle ΔABC, with right angle at B, and let α be the angle opposite a. Then, for any segment b, there is a right triangle ΔDEF, with right angle at E, having $\overline{DE} \cong b$ and $\angle EDF \cong \alpha$.*

Proof: Exercise. □

Definition 13.1. *Let a and b be two segments. By Lemma 13.8 there is a right triangle ΔABC with right angle at B and legs congruent to a and u. Let α be the angle opposite a.*



By Lemma 13.9 there is a right triangle ΔDEF, with right angle at E, $\overline{DE} \cong b$, and $\angle EDF \cong \alpha$. Then, we define the product ab of a and b to be the segment opposite α in ΔDEF.



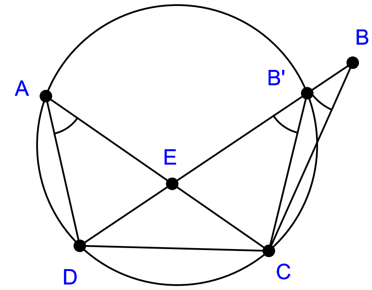
It is clear from all of the properties of segment order that we have covered, and using triangle congruence, that the product ab does not depend on the particular construction of the two right triangles in the definition. That is, any other construction of the triangles (as defined by the definition above) will yield a value of ab that is congruent to that of a given construction. Thus, ab is well-defined.

To show that segment multiplication has the right algebraic properties, we will need the following result on inscribed quadrilaterals.

Theorem 13.10. *Let $ABCD$ be a quadrilateral with B interior to $\angle CDA$. Then, $ABCD$ is an inscribed quadrilateral in a circle if and only if $\angle DAC \cong \angle DBC$.*

Proof: Suppose that $ABCD$ is inscribed in a circle c. Recall from Chapter 2 that $ABCD$ is *inscribed* in a circle if all of the points of $ABCD$ lie on the circle. Then Corollary 2.32 says that $\angle DAC \cong \angle DBC$ as these two angles both share the same arc on the circle.

Suppose that $\angle DAC \cong \angle DBC$. By Theorem 2.30 we can construct the circumscribed circle of ΔADC. Since B is interior to $\angle CDA$. we have, by the Crossbar Theorem, that $\overleftrightarrow{BD}$ intersects $\overline{AC}$ at some interior point E. Since all of the points interior to $\overline{AC}$ are inside the circle (Exercise), then E is inside the circle.



By circle continuity, we have that $\overleftrightarrow{BD}$ intersects the circle at some point B' that is on the same side of $\overleftrightarrow{CD}$ as A. By the first part of this theorem, we have $\angle DAC \cong \angle DB'C$ and thus $\angle DBC \cong \angle DB'C$. If $B' = B$ we are done. Otherwise, B is either outside or inside the circle. If B is outside, we get a contradiction to the Exterior Angle Theorem for $\Delta DB'C$. A similar argument shows that B cannot be inside the circle. □

In the proof of this theorem, we used several results from Chapter 2. A careful reading of the proofs of those theorems shows that they rely solely on theorems we have shown in the supplemental foundation chapters covered so far (Chapters 11, 12 and this chapter).

We are now ready to prove the algebraic properties of segment multiplication. We will use the short-hand $a = b$ to stand for $a \cong b$. This can be justified by the existence of the segment measure function.

Theorem 13.11. *Segment multiplication satisfies the following:*

(i) $au = a$ *for all segments* a.

(ii) $ab = ba$ *for all segments* a *and* b.

(iii) $a(bc) = (ab)c$ *for all segments* a, b, *and* c.

(iv) *For any segment* a, *there is a unique segment* b *such that* $ab = u$.

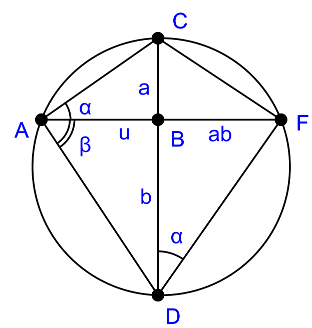
(v) $a(b + c) = ab + bc$ *for all segments* a, b, *and* c.

Proof: The proof of part (i) is left as an exercise.

For part (ii), we construct ab as follows. By Lemma 13.8 there is a right triangle ΔABC with right angle at B and legs congruent to a and

u. Let α be the angle opposite a. On the ray opposite $\overrightarrow{BC}$ from B we can find D such that $\overline{BD} \cong b$.

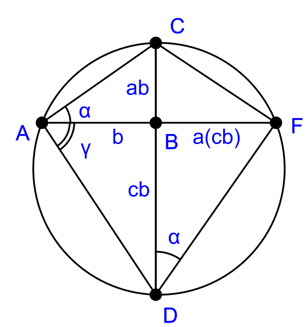
We can copy $\alpha = \angle BAC$ to the ray $\overrightarrow{DB}$ yielding an angle $\angle BDF$ with F on the opposite side of $\overleftrightarrow{DB}$ from A. Since $\angle BDF$ is not a right angle, then lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DF}$ must meet. We can assume F is this intersection point. Then ΔDBF is a right triangle and $\overline{BF}$ is the product ab.



Consider quadrilateral $ACFD$. Clearly, point D is interior to $\angle FCA$. Since $\angle FAC \cong \angle FDC$ then, by Theorem 13.10, we have that the quadrilateral is inscribed in a circle. Then, again by Theorem 13.10, we have that $\angle DAF \cong \angle DCF$. Then, since ΔABD has side u, we have that ΔCBF is the constructed triangle from the definition of segment multiplication. Thus, side $\overline{BF}$ is the segment defined as ba. We conclude that $ab = ba$

For part (iii), we start by constructing two right triangles, one with legs u and a and defining angle α. The other with legs u and c and defining angle γ. Then, we construct the triangle ΔABC defining ab.

We now copy the angle γ to $\overrightarrow{AB}$ getting $\angle BAD$ with D on the other side of $\overleftrightarrow{AB}$ from C. Then, $\overline{BD}$ will be cb. Next, we copy $\alpha = \angle BAC$ to the ray $\overrightarrow{DB}$ yielding an angle $\angle BDF$ with F on the opposite side of $\overleftrightarrow{DB}$ from A.

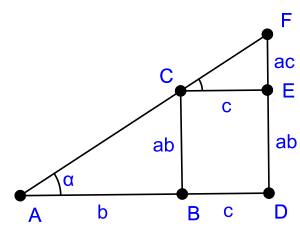


As in our proof of part (ii), we have that $\overleftrightarrow{AB}$ and $\overleftrightarrow{DF}$ must meet and we can assume F is this intersection point. ΔDBF is then a right triangle and $\overline{BF}$ is the product $a(cb)$. It is left as an exercise to show that $\overline{BF}$ is also $c(ab)$. By part (ii), we have $a(bc) = a(cb) = c(ab) = (ab)c$.

The proof of part (iv) is lefts as an exercise.

Part(v) is the distributive property. Construct a right triangle with legs u and a and defining angle α. Let ΔABC be the right triangle with leg b that defines ab.

On the ray opposite $\overrightarrow{BA}$ we can find D such that $\overline{BD} \cong c$. At D we construct the perpendicular to $\overleftrightarrow{BD}$. At C we construct the parallel $\overleftrightarrow{CE}$ to $\overleftrightarrow{BD}$. This intersects the perpendicular at a point, which we can assume is E. Then, $BCED$ is a parallelogram and $\overline{CE} \cong c$ by Theorem 13.5.



Since α is not a right angle, then $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$ intersect at a point F. Then, $\overleftrightarrow{AF}$ is a transversal crossing parallel lines and so $\angle ECF \cong \alpha$. Thus, $\overline{EF} \cong ac$.

Now, note that $\overline{AD}$ is the addition of b and c and $\overline{DF}$ is the addition of ab and ac. Also, $\overline{DF}$ is $a(b+c)$ by the definition of segment multiplication. Thus, $a(b+c) = ab + ac$. □

We will now use our new notion of segment multiplication to define triangle similarity. Similarity necessarily involves the use of proportions.

Definition 13.2. *Given four segments a, b, c, and d we say that these segments are* proportional *or* in the same ratio *if $ad = bc$. We notate this as $\frac{a}{b} = \frac{c}{d}$.*

Clearly, geometric proportions model the usual equality of fractions in the real numbers. We can now define similar triangles.

Definition 13.3. *Two triangles are* similar *if and only if there is some way to match vertices of one to the other such that corresponding sides are in the same ratio and corresponding angles are congruent.*

If ΔABC is similar to ΔXYZ, we shall use the notation $\Delta ABC \sim \Delta XYZ$. Thus, $\Delta ABC \sim \Delta XYZ$ if and only if

$$\frac{\overline{AB}}{\overline{XY}} = \frac{\overline{AC}}{\overline{XZ}} = \frac{\overline{BC}}{\overline{YZ}}$$

and

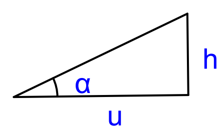
$$\angle BAC \cong \angle YXZ, \angle CBA \cong \angle ZYX, \angle ACB \cong \angle XZY$$

The major result for triangle similarity is the AAA similarity theorem. We start with a special case of right triangles.

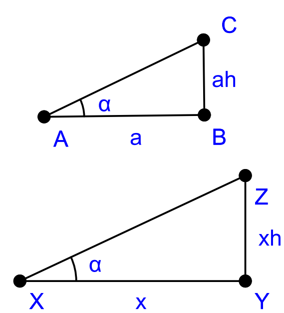
Theorem 13.12. *If in two right triangles there is a correspondence in which the two acute angles of one triangle are congruent to the two acute angles of the other triangle then the corresponding legs of the triangles are proportional.*

Proof:

Let α be one of the acute angles and consider a right triangle with angle α and adjacent leg being u. Let h be the other leg. We will use this triangle as the base for constructing segment products defined by the angle α.

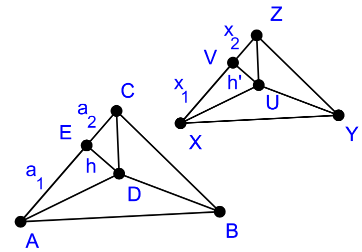


Now, suppose ΔABC and XYZ are the two right triangles with $\angle BAC$ and $\angle YXZ$ being α. Then, the other legs must have $\overline{BC} \cong ah$ and $\overline{YZ} \cong xh$, by the definition of segment multiplication. Then, $\overline{AB}\,\overline{YZ} = a(xh)$ and $\overline{BC}\,\overline{XY} = (ah)x$. Since $a(xh) = (ah)x$ by Theorem 13.11, we have that $\dfrac{\overline{AB}}{\overline{XY}} = \dfrac{\overline{BC}}{\overline{YZ}}$. $\square$



Theorem 13.13. *(AAA Similarity) If in two triangles there is a correspondence in which the three angles of one triangle are congruent to the three angles of the other triangle then the triangles are similar.*

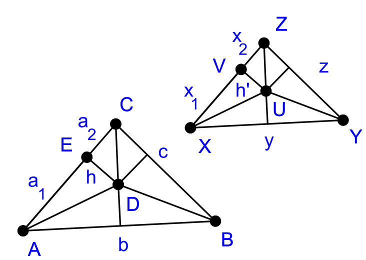
Proof: Let ΔABC and ΔXYZ be the two triangles. We start by constructing the angle bisectors of the three angles in both triangles. We saw in Project 2.2 that the angle bisectors of a triangle intersect at a common point called the *incenter* of the triangle. For ΔABC let D be the incenter and for ΔXYZ let U be the incenter.



Drop a perpendicular from D to $\overleftrightarrow{AC}$ intersecting at E. E must be interior to $\overline{AC}$, for if it were exterior, then we would have an exterior angle ($\angle DAC$) to ΔAED that is less than an interior angle, which contradicts the Exterior Angle Theorem. Let $h = \overline{DE}$, $a_1 = \overline{AE}$, and $a_2 = \overline{CE}$.

Likewise, we drop a perpendicular from U to $\overline{XZ}$ at V and let $h' = \overline{UV}$, $x_1 = \overline{VX}$, and $x_2 = \overline{VZ}$.

Since ΔCDE and ΔZUV are right triangles with corresponding congruent angles, then by Theorem 13.12 we have $a_2h' = x_2h$. Likewise, since ΔADE and ΔXUV are right triangles with corresponding congruent angles, we have $a_1h' = x_1h$. Thus,$(a_1 + a_2)h' = (x_1 + x_2)h$. Let $a = (a_1 + a_2)$ and $x = (x_1 + x_2)$. Then, $ah' = xh$.



We can also drop perpendiculars from D and U to the remaining sides of the triangle. By AAS triangle congruence we can show that, in each triangle, the segments created all are congruent to each other. (Exercise)

Let $b = \overline{AB}$, $c = \overline{BC}$, $y = \overline{XY}$, and $z = \overline{YZ}$. Then, by the argument given above, we have that $bh' = yh$ and $ch' = zh$. Also, $(ay)h = a(yh) = a(bh') = (ah')b = (xh)b = (bx)h$ by the properties of segment multiplication. By Theorem 13.11, part (iv), there is a segment h^{-1} such that $hh^{-1} = u$. Thus, $(ay)hh^{-1} = (bx)hh^{-1}$, or $(ay)u = (bx)u$. By part (i) of Theorem 13.11 we have $ay = bx$, or $\frac{a}{x} = \frac{b}{y}$. A similar argument shows the other two proportions. □

Exercise 13.2.1. *Prove Lemma 13.8.*

Exercise 13.2.2. *Prove Lemma 13.9.*

Exercise 13.2.3. *Given ΔABC, let c be the circumscribed circle of the triangle. Prove that any point interior to one of the sides of the triangle is inside the circumscribed circle. [Hint: Look at the construction used in the proof of Theorem 2.30.]*

Exercise 13.2.4. *Finish the proof of part (iii) of Theorem 13.11.*

Exercise 13.2.5. *Prove part (iv) of Theorem 13.11. [Hint: Construct two triangles in the appropriate way to get $ab = u$]*

Exercise 13.2.6. *Given segments a and b, if there is a segment x such that $xa = b$, we calll x the* division *of b by a, denoted by $x = \frac{b}{a}$. Provide a construction for x.*

Exercise 13.2.7. *Given two segments a and b, show that if $aa = bb$, then $a = b$.*

Exercise 13.2.8. *Given ΔABC, let I be the incenter. Drop a perpendicular to each side of the triangle creating points D, E, and F. Show that $\overline{ID} \cong \overline{IE} \cong \overline{IF}$. [Hint: Use AAS triangle congruence.]*

Exercise 13.2.9. *Given ΔABC, let B' and C' be points on $\overline{AB}$ and $\overline{AC}$ respectively. Suppose that $\overleftrightarrow{B'C'}$ is parallel to $\overleftrightarrow{BC}$. Show that $\overline{AB}$ and $\overline{AC}$ are proportional to $\overline{AB'}$ and $\overline{AC'}$.*

13.3 AREA

In Project 2.4 we gave a brief development of the concept of area in Euclidean geometry. Euclid's original development of area starts with Proposition 35.

> Parallelograms that are on the same base and in the same parallels are equal to one another.

Note that the proposition does not explicitly mention area. Instead we get a vague reference to "equality". Also, the setting for the parallelograms is not specific. A more precise version would be:

Theorem 13.14. *Let $ABCD$ and $EBCF$ be two parallelograms that share base $\overline{BC}$ and have $\overleftrightarrow{AF}$ parallel to $\overleftrightarrow{BC}$. Then, the two parallelograms are equal.*

As with many of Euclid's statements, the concept of two shapes or "figures" being "equal" is never defined exactly. However, from looking at Euclid's proofs, we can see that this type of equality has to do with splitting up the figures into pairs of congruent triangles. As in Project 2.4, we will say that the two figures are equally "decomposable." Before we can define area, we will need to investigate this notion of figures and decomposition in more detail. We start with precise definitions of terms. The material is this section is modeled after the discussion in Chapter 5 of [11].

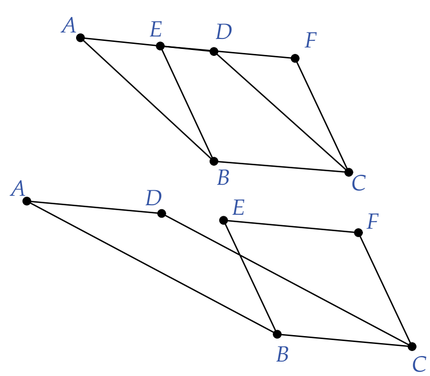
Definition 13.4. *A* figure *in the Euclidean plane is a subset of the plane that can be expressed as the finite union of non-overlapping triangles. Two triangles are* non-overlapping *if they have no interior points in common.*

Definition 13.5. *We will call two figures P and P'* equidecomposable *or* congruent by addition *if it is possible to decompose each into a finite union of non-overlapping triangles*

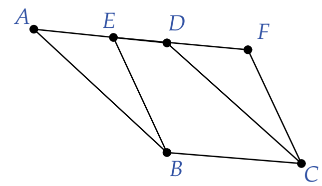
$$\begin{aligned} P &= T_1 \cup \cdots \cup T_n \\ P' &= T_1' \cup \cdots \cup T_n' \end{aligned}$$

where for each i, triangle T_i is congruent to T_i' [11, page 197].

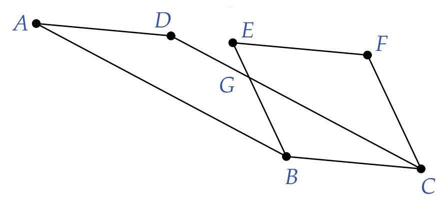
For Proposition 35, we first note that A, E, D, and F are all collinear. This is easily seen by using Theorem 13.2. Thus, there are two possible configurations for the parallelograms: either D will be between E and F as shown in the top configuration, or D will not be on segment $\overline{EF}$ as shown in the second configuration.



In the first case, where D is between E and F, it is left as an exercise to show that these two parallelograms are congruent by addition.



In the second case it is not apparent how one can decompose these figures into congruent triangles. For this case we will define an equivalence of figures by *subtraction.*



Definition 13.6. *We say that two geometric figures P and P' have* equal content *or are* congruent by subtraction *if there are figures Q, Q', that are congruent by addition with P and Q non-overlapping and P', Q' non-overlapping such that*

$$\begin{aligned} R &= P \cup Q \\ R' &= P' \cup Q' \end{aligned}$$

are congruent by addition.

It can then be shown that the two parallelograms in the second case described above are congruent by subtraction. (Exercise).

It is interesting to compare our two types of "equality" for figures. If two figures are congruent by addition, then they are congruent by subtraction (Q can be empty). But, the converse is not readily obvious. In fact, there are geometries in which the converse does not hold. These geometries are based on algebraic structures called *ordered fields* and are beyond the scope of this text. They satisfy most of the properties of Neutral geometry, except for the Archimedean property. (For a full development of these "weird" geometries, see sections 14-18 of [11].

If we take the axiomatic system we have developed so far, which includes the Archimedean property, and assume a reasonable method of assigning numerical areas to figures, then the converse is true. In any event, we will say that two figures are *equal* if they are congruent by addition or subtraction.

Proposition 36 is quite similar to Proposition 35.

Theorem 13.15. *(Proposition 36) Let $ABCD$ and $EFGH$ be two parallelograms with $\overline{AB} \cong \overline{EF}$, A, B, E, F collinear, and C, D, G, H collinear, then the parallelograms are equal (i.e., congruent by addition or subtraction).*

Proof: If $\overline{AB} = \overline{EF}$, then the result follows from Theorem 13.14.

Suppose the two parallelograms share an endpoint between $\overline{AB}$ and $\overline{EF}$. Without loss of generality, assume that $B = E$. The proof of this case is left as an exercise.

So, we assume that A and B do not match either E or F. By congruence axiom III-4, we can copy $\angle FEH$ to the ray $\overrightarrow{AB}$ yielding $\angle BAJ$ (Figure 13.3). Then, $\overleftrightarrow{AJ}$ must intersect $\overleftrightarrow{CD}$ at some point, for if these lines did not intersect, we would have two lines parallel to a third through a point, which contradicts Playfair's axiom. We can assume J is the point of intersection. Likewise, we can copy $\angle EFG$ to $\overrightarrow{BA}$ yielding $\angle ABK$ with K on $\overleftrightarrow{CD}$.

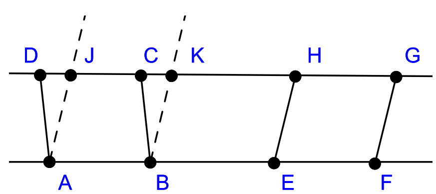


Figure 13.3

Since $\overleftrightarrow{EH}$ and $\overleftrightarrow{FG}$ are parallel, then, by Theorem 13.1 we have that $\angle EFG$ and $\angle FEH$ are supplementary. Since $\angle ABK \cong \angle EFG$ and $\angle BAJ \cong \angle FEH$, then, $\angle ABK$ and $\angle BAJ$ are supplementary. By Theorem 12.13, $\overleftrightarrow{AJ}$ is then parallel to $\overleftrightarrow{BK}$. Thus, $ABKJ$ is a parallelogram. By Theorem 13.5 we have that $\overline{AB} \cong \overline{JK}$, $\overline{AJ} \cong \overline{BK}$, and the opposite angles in $ABKJ$ are congruent.

Since $\angle BAJ \cong \angle FEH$, and $A \neq E$, we have that $\overleftrightarrow{AJ}$ is parallel to $\overleftrightarrow{EH}$, and thus $\overleftrightarrow{BK}$ is parallel to $\overleftrightarrow{EH}$ by Theorem 13.2. Now, $\overleftrightarrow{KE}$ is a traversal crossing the parallel lines $\overleftrightarrow{BK}$ and $\overleftrightarrow{EH}$. By Theorem 13.1, we have $\angle EKH \cong \angle KEB$ and $\angle BKE \cong \angle HEK$. Then, by ASA we have $\Delta EBK \cong \Delta KHE$. Thus, $\overline{BK} \cong \overline{HE}$. We conclude that $ABKJ \cong EFGH$.

Since $ABCD$ and $ABKJ$ share the same base, then these two parallelograms are congruent by addition or subtraction, by Theorem 13.14. The triangles used in the decomposition of $ABKJ$ can be put into a 1-1 correspondence with corresponding triangles in $EFGH$ that are congruent. Thus, it is clear that we can make $ABCD$ and $EFGH$ equal. □

Euclid's Propositions 37 and 38 deal with triangles defined between parallels. Before we consider those results, we need to dive deeper into the *congruent by addition* property. As mentioned above, we can consider this property as equivalent with congruent by subtraction if we assume our geometry is Archimedean. We first define the notion of an *equivalence relation*.

Definition 13.7. *An* equivalence relation *on a set S is a pairing, denoted by R, of elements from S that is:*

- **Reflexive:** *aRa for all a in S.*
- **Symmetric:** *aRb then bRa for all a, b in S.*
- **Transitive:** *If aRb and bRc, then aRc for all a, b, and c in R.*

An example of an equivalence relation would be the relation of congruence on the set S of all segments. We define aRb to mean that $a \cong b$ for segments a and b. This clearly satisfies all three conditions above.

Let S be the set of all figures in the plane, where figure is defined as in definition 13.4. Let f and f' be two figures. Define a relation R by fRf' if f and f' are congruent by addition. Clearly, this relation is reflexive and symmetric. It is also transitive.

Theorem 13.16. *The relation defined by the "congruent by addition" property is a transitive relation.*

Proof: Suppose that f and f' are congruent by addition and also that f' and f'' are congruent by addition. Then, f and f' can be written as the non-overlapping union of triangles:

$$f = T_1 \cup T_2 \cup \cdots \cup T_n$$
$$f' = T_1' \cup T_2' \cup \cdots \cup T_n'$$

where $T_i \cong T_i'$ for all i. Likewise, for f' and f'':

$$f' = S_1' \cup S_2' \cup \cdots \cup S_m'$$
$$f'' = S_1'' \cup S_2'' \cup \cdots \cup S_m''$$

We will show that we can find a decomposition of f and f'' into congruent triangles, based on the two decompositions above.

For each i and j, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$, consider the intersection $T_i' \cap S_j'$. If this intersection has an empty interior (is a point or a segment) we discard it. Otherwise, since the intersection of two triangles can be decomposed into non-overlapping triangles (not hard to prove, but tedious), we have

$$T_i' \cap S_j' = U_{ij1}' \cap U_{ij2}' \cap \cdots \cap U_{ijt}'$$

By Theorem 11.71, there is a congruence transformation g_i taking T_i to T_i' for each i. By exercise 11.11.2 we know that g_i^{-1} is a congruence transformation. Define

$$U_{ijk} = g^{-1}(U_{ijk}'), \;\; k = 1, \ldots, t$$

The U_{ijk} provide a non-overlapping set of triangles making up f. There is also a congruence transformation h_j taking S_j' to S_j'' for each j. Define

$$U_{ijk}'' = h(U_{ijk}'), \;\; k = 1, \ldots, t$$

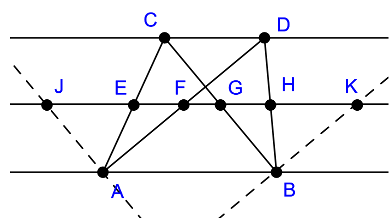
The U_{ijk}'' provide a non-overlapping set of triangles making up f''. Now, $U_{ijk} \cong U_{ijk}''$ for all k. We conclude that f and f'' are congruent by addition. □

We now consider Euclid's Propositions 37 and 38 on triangles in parallels.

Theorem 13.17. *(Proposition 37) Let ΔABC and ΔABD be two triangles that share base $\overline{AB}$ and have $\overleftrightarrow{CD}$ parallel to $\overleftrightarrow{AB}$. Then, the two triangles are equal.*

Proof:

Let E be the midpoint of $\overline{AC}$. By Theorem 13.7, the parallel to $\overleftrightarrow{AB}$ through E will intersect $\overline{AD}$ at F, $\overline{BC}$ at G and $\overline{BD}$ at H. At A construct the parallel to $\overleftrightarrow{BC}$. This will meet $\overleftrightarrow{EF}$ at some point J.



Likewise, construct the parallel to $\overleftrightarrow{AD}$ through B, meeting $\overleftrightarrow{EF}$ at K. With the transversal $\overleftrightarrow{AC}$ crossing parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AJ}$, we have that $\angle ECG \cong \angle EJA$. also, the vertical angles $\angle AEJ$ and $\angle GEC$ are congruent. Since E is the midpoint of $\overline{AC}$ we have by ASA that $\Delta JEA \cong \Delta GEC$. Thus, parallelogram $ABGJ$ is congruent by addition to ΔABC.

Similarly, we can show that parallelogram $ABKF$ is congruent by addition to ΔADB. By Theorem 13.14, we know that parallelograms $ABGJ$ and $ABKF$ are congruent by addition. Since congruence by addition is an equivalence relation, and thus transitive, we have that ΔABC is congruent by addition to ΔADB. □

Theorem 13.18. *(Proposition 38) Let ΔABC and ΔDEF be two triangles with $\overline{AB} \cong \overline{DE}$, A, B, D, E collinear, and $\overleftrightarrow{CF}$ parallel to $\overleftrightarrow{AB}$, then the triangles are equal.*

Proof: The proof is similar to that of Theorem 13.15 and is left as an exercise. □

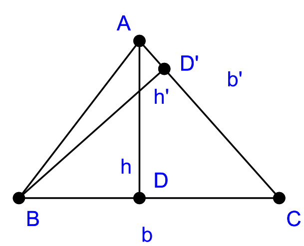
The title of this section is "Area." We are finally to the point where we can give a solid definition of area. To do so, we will use the properties of segment multiplication and similar triangles from the previous section.

Definition 13.8. *Given ΔABC let $b = \overline{AB}$. This will be a* base *of the triangle. From C drop a perpendicular to $\overleftrightarrow{AB}$ intersecting at D. Let $h = \overline{CD}$. This will be a* height *of the triangle corresponding to b. The* area *of the triangle is defined as $\frac{1}{2}bh$.*

Is area a well-defined notion? The quantity $\frac{1}{2}bh$ is a dyadic segment. These were defined and analyzed in Chapter 11. Is the definition independent of b and h?

Lemma 13.19. *Let ΔABC be a triangle. Let $b = \overline{AB}$ and let h be the corresponding height. Also, let $b' = \overline{AC}$ and let h' be the corresponding height. Then, $\frac{1}{2}bh = \frac{1}{2}b'h'$.*

Proof:



Let D be the point on $\overline{AB}$ defining the height h and let D' be the point on $\overline{AC}$ defining h'. Then, right triangles ΔADC and $\Delta BD'C$ share $\angle ACB$ so these triangles are similar. By AAA similarity we have $\frac{h}{b'} = \frac{h'}{b}$. That is, $bh = b'h'$. By dyadic arithmetic, $\frac{1}{2}bh = \frac{1}{2}b'h'$. □

We now want to extend our notion of area to more general shapes. The reasonable category of shapes to pick are those we have already reviewed carefully —figures.

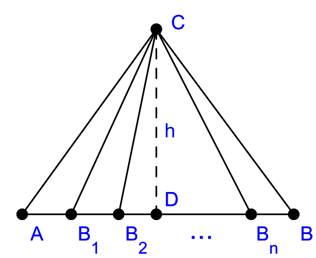
Definition 13.9. *Let f be a figure. By definition, f can be written as the union of a finite number of non-overlapping triangles, $f = T_1 \cup T_2 \cup \cdots \cup T_n$. Define the area of f to be the sum of the areas of these triangles.*

As with triangle area, the crucial question to ask is whether this definition is well-defined. We must show that the definition is independent of the decomposition of f into triangles. We start by proving this property for triangles in increasingly general cases.

Lemma 13.20. *Given ΔABC, suppose that the triangle is subdivided into the union of non-overlapping sub-triangles $T_i = \Delta ACB_i$, $i = 1, \ldots, n$, where B_i are distinct interior points on $\overline{AB}$. Then the sum of the areas of the sub-triangles is equal to the area of ΔABC.*

Proof:

From C drop a perpendicular to $\overleftrightarrow{AB}$ at D. Then $h = \overline{CD}$ will be a height for all of the sub-triangles. We can use betweenness to order the points such that $A * B_1 * B_2, B_1 * B_2 * B_3, \ldots, B_{n-1} * B_n * B$.

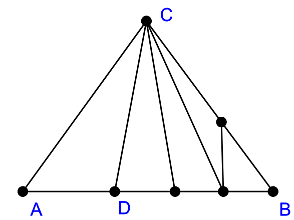


Thus, the sum of the areas for the sub-triangles will be $\frac{1}{2}\overline{AB_1}h + \frac{1}{2}\overline{B_1B_2}h + \cdots \frac{1}{2}\overline{B_nB}h$. By segment addition, this is equivalent to $\frac{1}{2}(\overline{AB_1} + \overline{B_1B_2} + \cdots \overline{B_nB})h = \frac{1}{2}\overline{AB}h$. □

Lemma 13.21. *Given ΔABC, suppose that the triangle is subdivided into the union of non-overlapping sub-triangles T_i, $i = 1, \ldots, n$, where all vertices of the sub-triangles, other than A, B, or C, are either on $\overline{AB}$ or $\overline{BC}$. Then the sum of the areas of the sub-triangles is equal to the area of ΔABC.*

Proof: We prove this by induction on n, the number of sub-triangles. If $n = 1$, the result follows from the previous lemma. Suppose the lemma is true for $n = k$ and suppose we have a triangle with $k+1$ sub-triangles with vertices as stipulated in the lemma.

Then, since there are no vertices of sub-triangles on $\overline{AC}$, then $\overline{AC}$ must be a side of one of the sub-triangles, say ΔACD. We can assume, without loss of generality, that D is on $\overline{AB}$.



Then, ΔBCD is sub-divided into k sub-triangles with no new vertex interior to $\overline{CD}$, as $\overline{CD}$ has all interior points that are interior to ΔABC.

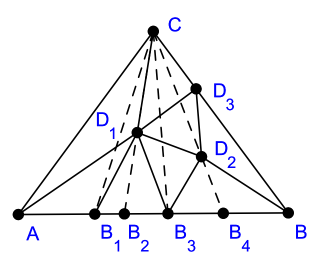
By the induction hypothesis, the area of ΔBCD is equal to the sum of the areas of these sub-triangles. By Lemma 13.20, the area of ΔABC is equal to the sum of the areas of ΔACD and ΔBCD. Thus, the area of ΔABC is equal to the sum of the areas of all of the original sub-triangles. □

We can now prove the general case. For the sake of brevity, we will let $a(\Delta ABC)$ stand for the area of ΔABC.

Theorem 13.22. *Given ΔABC, suppose that the triangle is sub-divided into the union of non-overlapping sub-triangles T_i, $i = 1, \ldots, n$. Then the sum of the areas of these sub-triangles is equal to the area of ΔABC.*

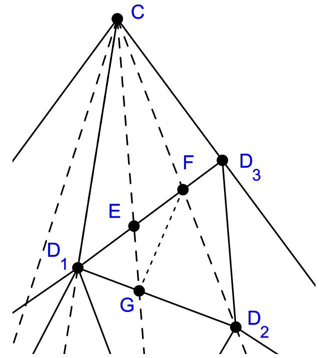
Proof: From C construct rays $\overrightarrow{CD_i}$, $i = 1, \ldots, n$, where D_i are the vertices create by the T_i's, other than A, B, or C.

For each D_i create rays $\overrightarrow{CD_i}$. By the Crossbar Theorem, these rays will intersect $\overline{AB}$. Also, there will be points on $\overline{AB}$ defined by vertices of the triangulation. Let $B_j, j = 1, \ldots, m$ denote the set of vertices from both of these sets, ordered by betweenness. (We illustrate this with $n = 3$ points D_i.)



Let $S_j = \Delta B_{j-1}CB_j, j = 1, \ldots m$, with $B_0 = A, B_m = B$. By Lemma 13.20, we have that $a(\Delta ABC) = \Sigma_{j=1}^{m} a(S_j)$.

Consider the intersections $T_i \cap S_j$. These intersections will either by triangles or quadrilaterals. As an example, here we have blown up the sub-triangle $T_i = D_1D_2D_3$ from the figure above. The rays $\overrightarrow{CB_3}$ and $\overrightarrow{CB_4}$ intersect $\overline{D_1D_3}$ at E and F. This creates ΔD_2D_3F, quadrilateral $GEFD_2$ and ΔD_1EG. By adding in $\overline{FG}$ we create a triangulation of the quadrilateral.



We conclude that we can create a refinement of the original triangulation by a new triangulation

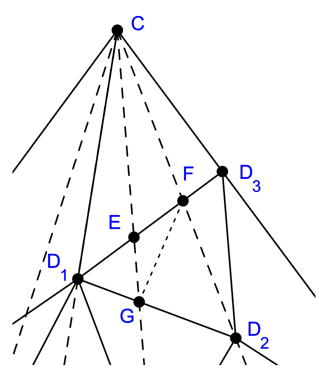
$$T_i \cap S_j = \Sigma_{k=1}^{t} U_{ijk}$$

where the U_{ijk} are directly defined by triangles in $T_i \cap S_j$, or possible triangles created from quadrilaterals in $T_i \cap S_j$.

For each j, the triangles U_{ijk} in this refinement that are in S_j will create a triangulation of S_j with no interior vertices, since the rays $\overrightarrow{CB_j}$ intersect all interior points of the original triangulation of ΔABC. By Lemma 13.21, we know that the area of each S_j is the sum of the areas of the U_{ijk} in the refinement. Since $a(\Delta ABC) = \Sigma_{j=1}^{m} a(S_j)$, we then have that $a(\Delta ABC) = \Sigma_{i,j,k}\, a(U_{ijk})$.

Now, we will show that this sum of areas of the U_{ijk}'s also equals the area found in the original triangulation. Let's review how the U_{ijk}'s were constructed.

Each intersection $T_i \cap S_j$ consist of triangles and quadrilaterals. Exactly one ray, here $\overrightarrow{CF}$, intersects $T_i = \Delta D_1D_2D_3$ at a vertex (D_2). This ray divides T_i into two sub-triangles: ΔD_1FD_2 and ΔD_2FD_3. Other rays may divide each of these two triangles into quadrilaterals or triangles, but we can add lines to create complete triangulations. The sub-triangles thus created are the U_{ijk}'s.



By Lemma 13.20, $a(T_i) = a(\Delta D_1 F D_2) + a(\Delta D_2 F D_3)$. By Lemma 13.21, $a(\Delta D_1 F D_2) = \Sigma a(U_{i'j'k'})$ and $a(\Delta D_2 F D_3) = \Sigma a(U_{i''j''k''})$, for some subsets of U_{ijk}'s from the refinement $T_i \cap S_j = \Sigma_{k=1}^{t} U_{ijk}$. Thus, $a(T_i) = \Sigma a(U_{i'j'k'}) + \Sigma a(U_{i''j''k''})$. Then, since each U_{ijk} appears in exactly one T_i, we have that

$$a(\Delta ABC) = \Sigma_{i,j,k}\, a(U_{ijk}) = \Sigma_i a(T_i)$$

□

We have now shown that Definition 13.9 is well-defined for triangles. It remains to show this for general figures.

Theorem 13.23. *Let f be a figure. Suppose f can be written as the union of a finite number of non-overlapping triangles, $f = T_1 \cup T_2 \cup \cdots \cup T_n$ and also as the sum of a finite number of non-overlapping triangles, $f = T_1' \cup T_2' \cup \cdots \cup T_m'$. Then, $\Sigma_{i=1}^{n} a(T_i) = \Sigma_{j=1}^{m} a(T_i')$.*

Proof: As in the proof of Theorem 13.16, we can find a triangulation of each intersection $T_i \cap T_j'$ such that $T_i \cap T_j' = \Sigma_k\, U_{ijk}$, for sub-triangles U_{ijk}. By Lemma 13.22, $a(T_i) = \Sigma_{j,k}\, a(U_{ijk})$ and $a(T_j') = \Sigma_{i,k}\, a(U_{ijk})$. Thus, $\Sigma_{i=1}^{n}\, a(T_i) = \Sigma_{i,j,k}\, a(U_{ijk}) = \Sigma_{j=1}^{n}\, a(T_j')$. □

Now that we have shown that area is a well-defined concept, let's check that it satisfies the fundamental properties that we intuitively think of for area.

Theorem 13.24. *For the area function defined in 13.8 and 13.9 we have*

1. *For any triangle T, we have $a(T) > 0$.*
2. *If triangles T and T' are congruent, then $a(T) = a(T')$.*
3. *If two figures f and f' have an empty intersection, then $a(f \cup f') = a(f) + a(f')$.*
4. *If f is a figure, then $a(f) > 0$.*
5. *If figures f and f' are equal (congruent by addition), then $a(f) = a(f')$.*

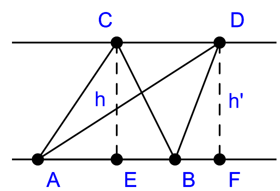
Proof: The proof is left as an exercise. □

We now return to our review of Euclid's propositions. Propositions 39 and 40 are the converse statements to Propositions 37 and 38.

Theorem 13.25. *(Proposition 39) Let ΔABC and ΔABD be two triangles that share base $\overline{AB}$ and are equal (congruent by addition). Then, $\overleftrightarrow{CD}$ is parallel to $\overleftrightarrow{AB}$.*

Proof: By Theorem 13.24, part (v), we know that the two triangles have the same area. Drop perpendiculars from C to $\overleftrightarrow{AB}$ at E and from D to $\overleftrightarrow{AB}$ at F. Then, $h = \overline{CE}$ is the height of ΔABC and $h' = \overline{DF}$ is the height of ΔABD. Also, $a(\Delta ABC) = \frac{1}{2}ah$ and $a(\Delta ABD) = \frac{1}{2}ah'$, and so we have $\frac{1}{2}ah = \frac{1}{2}ah'$. By the properties of dyadic segments, $h = h'$.

Since the angles at E and F are right angles, then $\overleftrightarrow{CE}$ and $\overleftrightarrow{DF}$ are parallel, by Theorem 12.11 and so $CDFE$ is a parallelogram (exercise). Thus, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ are parallel. The result then follows from Theorem 13.17. □



Theorem 13.26. *(Proposition 40) Let ΔABC and ΔDEF be two triangles with $\overline{AB} \cong \overline{DE}$, A, B, D, E collinear. If the triangles are equal (congruent by addition), then $\overleftrightarrow{CF}$ is parallel to $\overleftrightarrow{AB}$.*

Proof: The proof is left as an exercise. □

Proposition 41 compares areas of parallelogram and triangles.

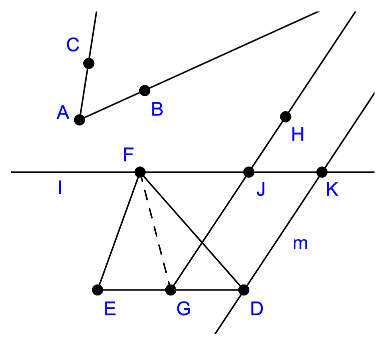
Theorem 13.27. *(Proposition 41) Let ΔABC and parallelogram $ABDE$ be defined with C, D, and E collinear and $\overleftrightarrow{CD}$ parallel to $\overleftrightarrow{AB}$. Then the area of ΔABC is half that of parallelogram $ABDE$.*

Proof: The proof is left as an exercise. □

Proposition 42 is a construction proposition.

> **Theorem 13.28.** *(Proposition 42) It is possible to construct a parallelogram, with a specified angle, having the same area as a given triangle.*

Proof: Let the given angle be $\angle BAC$ and the given triangle be ΔDEF. Let G be the midpoint of $\overline{DE}$. On $\overrightarrow{GD}$ we can create $\angle DGH$ congruent to $\angle BAC$ with H and F on the same side of $\overleftrightarrow{AB}$.



Let l be the parallel to $\overleftrightarrow{DE}$ through F. Then, since transversal $\overleftrightarrow{GH}$ crosses $\overleftrightarrow{GD}$, it must cross l at some point J. Let m be the parallel to $\overleftrightarrow{GJ}$ at D. Then, m is a transversal to parallel lines l and $\overleftrightarrow{GD}$, so m crosses ł at some point K. Then, $GJKD$ is a parallelogram.

Now, $\overline{AE} \cong \overline{EB}$ for ΔACE and ΔECB. Also, A, B, E are collinear, and l is parallel to $\overleftrightarrow{DE}$. Thus, by Theorem 13.18 we have that ΔACE and ΔECB are equal, and thus by Theorem 13.24 the triangles have the same area. Thus, by additivity of area, $a(\Delta GDF) = \frac{1}{2}a(\Delta EDF)$.

Now, ΔGDF and $GJKD$ have the same height, as they are defined between the same parallels (Exercise). By Theorem 13.27, we have that $a(\Delta GDF) = \frac{1}{2}a(GJKD)$. Thus, $a(\Delta EDF) = a(GJKD)$. □

Proposition 43 is a somewhat strange construction theorem. It's utility is mainly found in the proof of later propositions.

> **Theorem 13.29.** *(Proposition 43) Given parallelogram $ABCD$ and its diameter $\overline{AC}$, let K be a point on the diameter. Draw parallels $\overline{EF}$ to $\overline{BC}$ and $\overline{GH}$ to $\overline{AB}$, both through K (Figure 13.4). Then, complements $EKBG$ and $HKFD$ are parallelogram with the same area.*

Proof: We first note that, by construction, $EKBG$, $HKFD$, $AEKH$, and $GKFC$ are all parallelograms. Thus, we have many transversals and many angle congruences.

For example, $\angle EAK \cong \angle HKA$ by transversal $\overleftrightarrow{AK}$ crossing parallels $\overleftrightarrow{AE}$ and $\overleftrightarrow{HK}$. Similarly, we get that $\angle KAH \cong \angle AKE$ from transversal

$\overleftrightarrow{AK}$ crossing parallels $\overleftrightarrow{AH}$ and $\overleftrightarrow{EK}$. By ASA triangle congruence we have $\Delta AEK \cong \Delta KHA$.

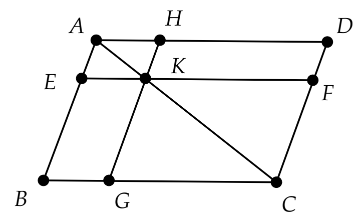


Figure 13.4

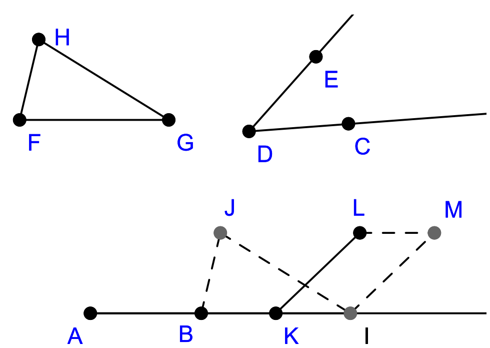
Similarly, we have $\Delta GKC \cong \Delta FCK$. By Theorem 13.5 we know that $\Delta ABC \cong CDA$. By additivity of area, we have that $a(\Delta ABC) = a(\Delta AEK) + a(EKBG) + a(GKC)$ and $a(\Delta CDA) = a(\Delta KHA) + a(HKFD) + a(FCK)$. Since congruent triangles have the same area, then $a(\Delta ABC) = a(\Delta AEK) + a(HKFD) + a(GKC)$. By the arithmetic of dyadic segments, we conclude that $a(EKBG) = a(HKFD)$. □

Proposition 44 says that we can construct a parallelogram of equal area to a triangle in a very general configuration.

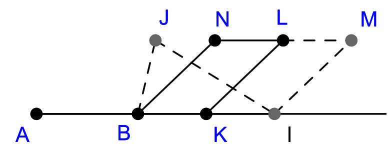
Theorem 13.30. *(Proposition 44) It is possible to construct a parallelogram on a given segment, with a given angle, having the same area as a given triangle.*

Proof:

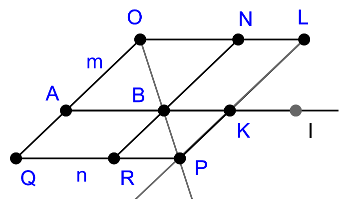
Let the given segment be $\overline{AB}$, the given angle be $\angle CDE$ and the given triangle be ΔFGH. On the ray opposite to $\overrightarrow{BA}$ we can find I such that $\overline{FG} \cong \overline{BI}$. With $\overline{BI}$ as base we can copy the angles in ΔFHG, yielding a congruent triangle ΔBIJ. Let K be the midpoint of $\overline{BI}$ and let $KIML$ be the parallelogram with area equal to the area of ΔFGH, as defined in the construction used in Theorem 13.28.



Let N be the intersection of the parallel to $\overleftrightarrow{KL}$, through B, with $\overleftrightarrow{LM}$. Then, $BKLN$ is a parallelogram. By Theorem 13.15 we have that $a(BKLN) = a(KIML)$, and so $a(BKLN) = a(\Delta FGH)$.



Let m be the parallel to $\overleftrightarrow{KL}$ through A. Then, $\overleftrightarrow{LN}$ is a transversal across parallel lines, so $\overleftrightarrow{LN}$ crosses l at some point O and $ABNO$ is a parallelogram.



Since $\overleftrightarrow{AO}$ and $\overleftrightarrow{KL}$ are parallel, then $\angle AON$ and $\angle OLK$ are supplementary, by Theorem 13.1. Since B is not on $\overrightarrow{OA}$, then $\angle AON$ is not congruent to $\angle BON$. Thus, $\angle BON$ and $\angle OLK$ are not supplementary. By the parallel postulate, $\overleftrightarrow{OB}$ and $\overleftrightarrow{KL}$ intersect at some point P.

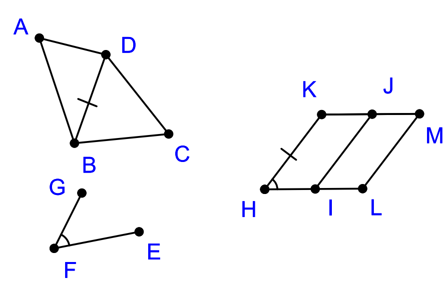
Let n be the parallel to $\overleftrightarrow{LN}$ through P. Then, n is a transversal to parallels m and $\overleftrightarrow{KL}$ and so n and m intersect at some point Q. We have thus constructed a parallelogram $OLPQ$ with point B on the diagonal $\overline{OP}$. By Theorem 13.29 we have that parallelogram $ABRQ$ has the same area as $BKLN$. We conclude that the area of $ABRQ$ is the same as the area of ΔFGH. □

Proposition 45 gives another parallelogram construction to match an area —in this case to match the area of a generic quadrilateral.

Theorem 13.31. *(Proposition 45) It is possible to construct a parallelogram with a given angle having the same area as a given quadrilateral.*

Proof:

The diagonal divides the quadrilateral into two triangles. Then one can use the previous theorem to construct parallelograms having area equal to the two triangles. The details to finish the proof are left as an exercise. □



Proposition 46 involves the construction of a square on a segment.

Theorem 13.32. *(Proposition 46) It is possible to construct a square on a given segment.*

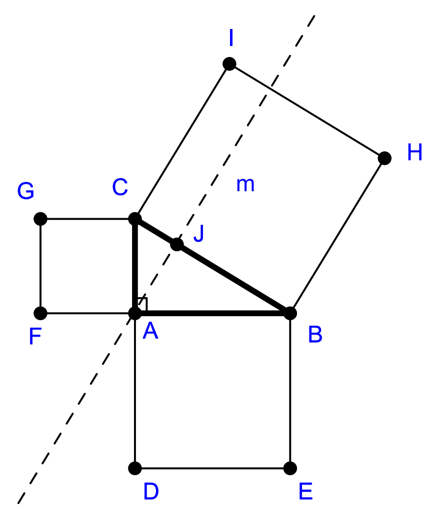
Proof: The proof is left as an exercise. □

We now come to the grand finale of Book I of Euclid's *Elements* —the Pythagorean Theorem and its converse.

Theorem 13.33. *(Pythagorean Theorem) In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the legs.*

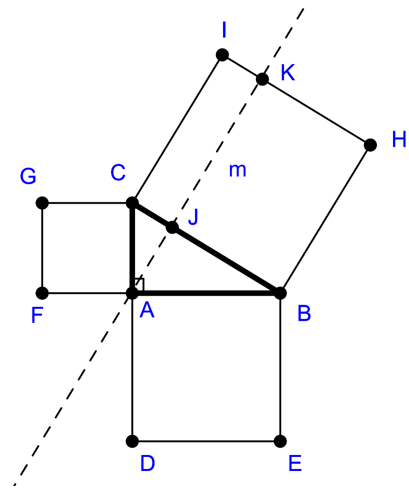
Proof: Let ΔBAC be the right triangle with right angle $\angle BAC$. The hypotenuse is by definition the side of the triangle opposite the right angle, so the hypotenuse is $\overline{BC}$. The legs are $\overline{AC}$ and $\overline{AB}$.

By Theorem 13.32 we can construct the squares on each of the sides of the triangle, as shown here. We can use betweenness to insure that these squares contain no point interior to the triangle. At A, construct the parallel m to $\overline{CI}$ in square $BCIH$. Since $\overleftrightarrow{CI}$ and $\overleftrightarrow{BC}$ are perpendicular, then m must intersect $\overleftrightarrow{BC}$ at some point J.

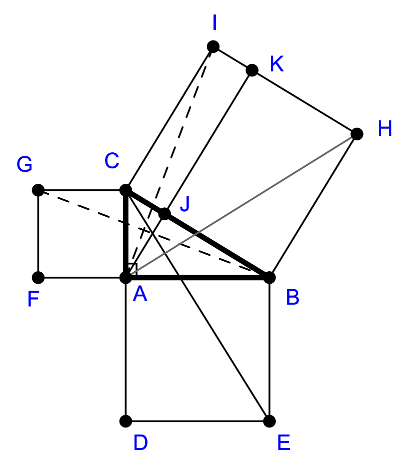


By Theorem 13.1, we know that $\overleftrightarrow{AJ}$ is perpendicular to $\overleftrightarrow{BC}$. We claim that J must be interior to $\overline{BC}$. If not, then either $B*C*J$ or $J*B*C$. Suppose $B*C*J$. Consider ΔAJC. This has a right angle at J. Also, since $\angle ACB$ is acute, then $\angle JCA$ would be greater than a right angle. But, this would contradict Theorem 13.3. Similarly, we cannot have $J*B*C$, and so J must be interior to $\overline{BC}$.

Since $\overline{BC} \cong \overline{HI}$, then there is K on $\overline{HI}$ such that $\overline{CJ} \cong \overline{IK}$. By SAS triangle congruence, $\Delta CIK \cong \Delta ICJ$ and since the angles at C and I in the square are right angles, we then have $\angle JCK \cong \angle KIJ$. By SAS again, we have $\Delta JCK \cong \Delta KIJ$, and the angles at J and K in $CJKI$ are congruent. Since the sum of the angles in a quadrilateral must be two right angles, we have that the angles at J and K in $CJKI$ are right angles, and $CJKI$ is a rectangle, as is $BHKJ$.



Consider ΔBGC and ΔIAC. We know that $\angle GCB$ is $\angle GCA$ plus $\angle ACB$ and $\angle ICA$ is $\angle ICJ$ plus $\angle ACB$. Since $\angle GCA$ and $\angle ICJ$ are congruent, then $\angle GCB \cong \angle ICA$. Also, $\overline{CI} \cong \overline{BC}$ and $\overline{AC} \cong \overline{CG}$. Thus, by SAS we have $\Delta GCB \cong \Delta ACI$. Similarly, we get $\Delta ECB \cong \Delta AHB$. Since $\angle CAF$ and $\angle BAC$ are right angles, then F, A, and B are collinear. Thus, $FGCA$ and ΔGCB are in the same parallels.



By Theorem 13.27, $a(\Delta GCB) = \frac{1}{2}a(FGCA)$. Since $\Delta GCB \cong \Delta ACI$, we have that $a(\Delta GCB) \cong a(\Delta ACI)$. Since $CIKJ$ and ΔACI are in the same parallels, we have that $a(\Delta ACI) = \frac{1}{2}a(CIKJ)$. We conclude that $a(FGCA) = a(CIKJ)$.

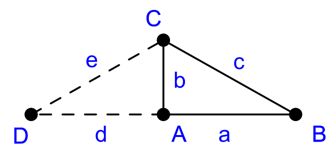
An exactly analogous argument shows that $a(ABED) = a(BHKJ)$. The result follows from additivity of area. □

The last proposition in Book I of the *Elements* is the converse to the Pythagorean Theorem.

Theorem 13.34. *(Proposition 48) In triangle ΔABC, if the square on $\overline{BC}$ is equal to the sum of the squares on the other sides, then the angle opposite $\overline{BC}$ is a right angle.*

Proof:

Let $a = \overline{AB}$, $b = \overline{AC}$, and $c = \overline{BC}$. Let $\overleftrightarrow{AD}$ be the perpendicular to $\overleftrightarrow{AC}$ at A. We can assume D has the property that $\overline{AD} \cong \overline{AB}$. Let $d = \overline{AD}$ and $e = \overline{CD}$.

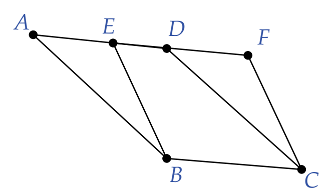


By Theorem 13.27, we know that the square constructed on segment a has area twice that of a triangle with height a. Thus, the area of the square constructed on segment a is aa.

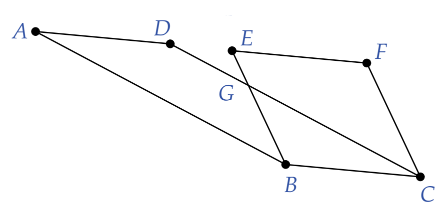
With this in mind, we can write the hypothesis of the theorem as $cc = aa + bb$. Since ΔDAC is a right triangle with hypotenuse e we have by the Pythagorean Theorem that $ee = bb + dd$. But, $d = a$, as these segments are congruent. Then, $dd = aa$, and by dyadic segment arithmetic we have $ee = cc$. By exercise 13.2.7 we have that $e = c$. By SSS triangle congruence, $\Delta ABC \cong \Delta ADC$. Thus, $\angle BAC$ is a right angle. □

Exercise 13.3.1. *Let l and m be two parallel lines. Let A and B be distinct points on l. Drop perpendiculars from A and B to m, yielding points C and D, respectively. Show that $\overline{AC} \cong \overline{BD}$.*

Exercise 13.3.2. *Prove the first case of Theorem 13.14, where D is between E and F.*



Exercise 13.3.3. *Prove the second case of Theorem 13.14, where D is not between E and F. [Hint: Let $Q = Q'$ be the figure defined by triangle DEG. Show that the two parallelograms are congruent by subtraction, using figure Q.]*



Exercise 13.3.4. *Finish the proof of Theorem 13.15. That is, given parallelograms $ABCD$ and $EFGH$ with $\overline{AB} \cong \overline{EF}$, A, B, E, F collinear, C, D, G, H collinear, and $B = E$, show that the parallelograms are equal.*

Exercise 13.3.5. *Prove Theorem 13.18.*

Exercise 13.3.6. *Prove the first three statements of Theorem 13.24.*

Exercise 13.3.7. *Prove the last two statements of Theorem 13.24.*

Exercise 13.3.8. *Let $CDFE$ be a quadrilateral with right angles at E and F and with $\overline{CE} \cong \overline{DF}$. Show that $CDFE$ is a parallelogram.*

Exercise 13.3.9. *Prove Theorem 13.26.*

Exercise 13.3.10. *Prove Theorem 13.27.*

Exercise 13.3.11. *Finish the proof of Theorem 13.31.*

Exercise 13.3.12. *Prove Theorem 13.32.*