CHAPTER 11

# Universal Foundations

> This present investigation is a new attempt to establish for geometry a *complete* and *as simple as possible*, set of axioms and to deduce from them the most important geometric theorems in such a way that the meaning of the various groups of axioms, as well as the significance of the conclusions that can be drawn from the individual axioms, comes to light.
>
> – David Hilbert in *Foundations of Geometry* [13] (1862–1943)

Throughout this text we have tried to do as Hilbert suggested in the introduction to his classic work on the foundations of geometry. There has been an emphasis on presenting the various strands of geometry in the most straight-forward and direct manner possible. We have worked to enhance our intuitive understanding of geometric concepts through the use of computer and group lab projects.

At the same time, we have tried to be as complete as possible in our exploration of geometric ideas by delving deeply into such topics as the complex function theory underlying a full development of the models of hyperbolic geometry.

What has been missing from this focus on completeness is a rigorous axiomatic treatment of the three major geometries covered in this text – Euclidean, Hyperbolic, and Elliptic. In Chapter 1 we saw that Euclid's original set of five axioms for Euclidean geometry were by no means complete. There were many hidden assumptions in Euclid's work that were never given a firm axiomatic basis. Among these assumptions were properties of continuity of geometric figures such as circles and lines, the unboundedness of the Euclidean plane, the transformations of figures, and even the existence of points.

Many attempts have been made to develop a more complete axiomatic basis for Euclidean geometry. We covered one such system, Birkhoff's system, in Chapter 3. This system was extremely economical, requiring only four axioms. It's elegance and power come from setting the foundations of geometry firmly on a model of analytic geometry, on the properties of the real numbers.

In this chapter we will look at an axiomatic system that is much more in the style of Euclid in that it is based on geometric rather than arithmetic foundations. Hilbert achieves an integration of the synthetic approach of classical geometry with the analytic approach of more modern systems of geometry. He does this by showing that the real numbers can be *constructed* geometrically. Hilbert's axiomatic development of geometry thus includes an axiomatic basis for analysis, and as such, is one of the great achievements of the modern era in mathematics.

Hilbert uses five groups of axioms to form the foundation of Euclidean geometry: axioms of incidence, betweenness, congruence, continuity, and parallelism. The first four groups of axioms can also serve as a foundational basis for Hyperbolic Geometry. With some tweaking of the betweenness axioms, the results of this chapter also form a solid foundation for Elliptic geometry. This will be covered in detail in Chapter 14.

## 11.1 INCIDENCE GEOMETRY

Perhaps the most obvious flaw in Euclid's set of axioms is his assumption that points and lines exist, and that points are incident on lines in the way we expect them to be. Hilbert's first four axioms are designed to put these intuitive notions on a firm footing.

Hilbert also realized that Euclid's definitions of a point as "that which has no part" and of a line as "breadth-less length" were basically meaningless, and thus did not define these two terms. Hilbert realized that what was important was how these undefined entities related to one another, their *relational* properties. These are the properties spelled out in the first five axioms of incidence.

- **I-1 Through any two distinct points $A$ and $B$ there is always a line $m$.**
- **I-2 Through any two distinct points $A$ and $B$, there is not more than one line $m$.**

- **I-3 On every line there exists at least two distinct points. There exist at least three points not all on the same line.**

- **I-4 Through any three points not on the same line, there is one and only one plane.**

Note that the idea of incidence itself is not defined in the axioms. This is another undefined term that is given relational properties. The precise definition of a point being *incident* on a line, or *lying on* a line, is not important. What is important is how the property of incidence is manifested by points and lines. These properties are spelled out in the axioms.

The first axiom is essentially the same as Euclid's first axiom in the *Elements*, although Euclid's axiom uses the language of "drawing" or "constructing", which is problematic. In Hilbert's formulation of this axiom, there is no reference to any physical action, only the connection of two points with a line.

The last axiom seems somewhat odd. Hilbert's original set of axioms were designed for three-dimensional geometry, with *plane* a third undefined geometric object. Hilbert developed an additional set of four axioms to cover incidence properties of planes. We will abbreviate Hilbert's set of incidence axioms to the four axioms above, which focus on planar geometry, and use the fourth axiom to ensure that when we speak of the "plane", there is a unique object of reference.

Hilbert's insistence on focusing on the relational properties of geometric objects, rather than their definitions, is evidenced by a famous adage attributed to him: "One must be able to say at all times —instead of points, lines and planes —tables, chairs, and beer mugs."

Let's see how these very basic axioms of incidence can be used to further our knowledge of lines and points. For our proofs we will assume basic rules of logical reasoning. For example, two things are either equal or not equal. Also, we will assume that proof techniques such as proof by contradiction are valid.

**Theorem 11.1.** *Given two distinct lines $l$ and $m$, they have at most one intersection point.*

Proof: Suppose that $l$ and $m$ intersected at two distinct points $A \neq B$. Then, through $A$ and $B$ we would have two distinct lines, which contradicts axiom I-2. Thus, they cannot intersect in two distinct points. □

**Theorem 11.2.** *For every line there is at least one point not on that line.*

Proof: Exercise. □

**Theorem 11.3.** *For every point there is at least one line not passing through it.*

Proof: Exercise. □

**Theorem 11.4.** *For every point there are at least two distinct lines that pass through it.*

Proof: Exercise. □

**Theorem 11.5.** *There exist three distinct lines such that no one point is on all three lines.*

Proof: Exercise. □

We note here that Hilbert's incidence axioms are suitable for Hyperbolic and Elliptic geometry, as incidence properties for all of the models we have considered for these geometries satisfy Hilbert's four incidence axioms.

## 11.2 BETWEENNESS GEOMETRY

The incidence axioms of the preceding section deal with issues of existence of lines and points.

The *betweenness* axioms deal with the ordering of points on a line. For this reason, they are often called the *axioms of order*. These axioms are necessary to make explicit what is meant by saying a point is "between" two other points on a line. Again, we do not specifically define what betweenness means. We instead provide axioms for how betweenness *works*.

- **II-1 If $B$ is a point between $A$ and $C$ (denoted $A * B * C$) then $A$, $B$, and $C$ are distinct points on the same line and $C * B * A$.**

- **II-2 For any distinct points $A$ and $C$, there is at least one point $B$ on the line through $A$ and $C$ such that $A * C * B$.**

- **II-3 If $A$ , $B$, and $C$ are three points on the same line, then exactly one is between the other two.**

- **II-4 (Pasch's Axiom) For this axiom, we need the following definition.**

  **Definition 11.1.** *The* segment $\overline{AB}$ *is defined as the set of points between A and B together with A and B.*

  **Let $A$, $B$, and $C$ be three non-collinear points and let $m$ be a line that does not contain any of these points. If $m$ contains a point of segment $\overline{AB}$, then it must also contain a point of either $\overline{AC}$ or $\overline{BC}$.**

We define a ray as follows:

**Definition 11.2.** *The* ray *from A through B is the set of points on segment* $\overline{AB}$ *together with points C such that* $A * B * C$.

Intuitively, axiom II-2 guarantees that the ray from $A$ through $B$ will be "bigger" than the segment $\overline{AB}$. This is essentially the same as Euclid's second axiom on extending lines.

Axiom II-3 guarantees that we have a well-defined ordering of points. It rules out the case of interpreting circles as lines, as we did in the development of Elliptic geometry. Given three equally-spaced points on a circle, any of the three points could be considered as *between* the other two. Thus, Hilbert's third axiom on betweenness fails for Elliptic geometry. In Chapter 14, we will define an alternative set of betweenness axioms, based on sets of four points, that will capture the notion of betweenness in Elliptic geometry.

Although the betweenness axioms cannot be used for Elliptic geometry, they are valid for all of our models of Hyperbolic geometry.

With rays and segments defined, we can describe the relationship of rays and segments to lines.

**Theorem 11.6.** *Given $A$ and $B$ we have :*

- $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$,
- $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

Proof:

- Clearly, by using the definitions of ray and segment we have that $\overline{AB} \subset \overrightarrow{AB}$ and $\overline{AB} \subset \overrightarrow{BA}$. So, $\overline{AB} \subset \overrightarrow{AB} \cap \overrightarrow{BA}$.

  On the other hand let $C$ be an element of this intersection. If $C$ is $A$ or $B$ then $C$ is clearly on $\overline{AB}$. Assume $C$ is not $A$ and not $B$. Since $C$ is on $\overrightarrow{AB}$, then it is either between $A$ and $B$ or satisfies $A * B * C$. Likewise, if $C$ is on $\overrightarrow{BA}$ then it is either between $A$ and $B$ or satisfies $C * A * B$. Since $C$ is on both rays then we must have by axiom II-3 that $C$ is between $A$ and $B$, and thus $C$ is on $\overline{AB}$. Thus, $\overrightarrow{AB} \cap \overrightarrow{BA} \subset \overline{AB}$.

  Since $\overline{AB} \subset \overrightarrow{AB} \cap \overrightarrow{BA}$ and $\overrightarrow{AB} \cap \overrightarrow{BA} \subset \overline{AB}$, then $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

- Let $C$ be an element of $\overrightarrow{AB} \cup \overrightarrow{BA}$. If $C = A$ or $C = B$ then, $C$ is certainly on the line $\overleftrightarrow{AB}$. Otherwise, suppose $C$ is on $\overrightarrow{AB}$ with $C$ not equal to $A$ or $B$. Then, either $A * C * B$ or $A * B * C$. But, this implies by axiom II-1 that $A, B, C$ are collinear. Likewise, if $C$ is on $\overrightarrow{BA}$, then $A, B, C$ are collinear. In either case, we get that $C$ is on $\overleftrightarrow{AB}$.

  On the other hand let $C$ be an element of $\overleftrightarrow{AB}$. By axiom II-3 we see immediately that $C$ is in $\overrightarrow{AB}$ or in $\overrightarrow{BA}$. Thus, $C$ is in $\overrightarrow{AB} \cup \overrightarrow{BA}$.

□

The next theorem guarantees that there is always a point between two distinct points.

**Theorem 11.7.** *Given points $A \neq B$ there is always a point $C$ with $A * C * B$. (see Figure 11.1)*

Proof: Note that this is slightly different than axiom II-2. Here we are saying that we can always find a point between two given points. In axiom II-2 we are saying that we can always find a point “outside” the given pair.

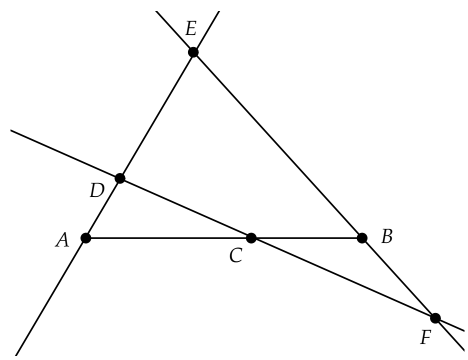



Figure 11.1

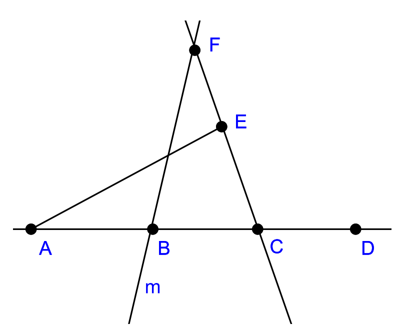
By Theorem 11.2 there is a point $D$ not collinear with $A$ and $B$. Axiom II-2 says that we can find a point $E$ with $A*D*E$. The incidence axioms guarantee the existence of the line through $E$ and $B$. Again, using axiom II-2 we can find a point $F$ on $\overleftrightarrow{EB}$ so that $E*B*F$. Now, $\overleftrightarrow{DF}$ contains a point $(D)$ of $\overline{AE}$. Also, $\overleftrightarrow{DF}$ intersects $\overleftrightarrow{EB}$ at $F$, which is outside $\overline{EB}$ and cannot intersect this line more than once. Thus, by axiom II-4 we have that $\overleftrightarrow{DF}$ must intersect $\overline{AB}$ at some point $C$. By the definition of a segment, $A*C*B$. □

The next two theorems deal with the relations between four points on a line. These results are so important that we will henceforth call them the *four-point properties*.

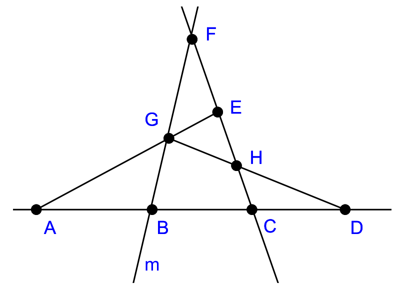
**Theorem 11.8.** *Let $A$, $B$, $C$, and $D$ be distinct points on a line $l$. If $B$ is on $\overline{AC}$ and $C$ is on $\overline{BD}$, then $B$ and $C$ are both on $\overline{AD}$. (That is, if $A*B*C$ and $B*C*D$ then $A*B*D$ and $A*C*D$)*

Proof:

By Theorem 11.2 there must be a point $E$ that is not on $l$. By axiom II-2 we can find a point $F$ such that $C*E*F$. Construct segment $\overline{AE}$ and let $m$ be the line $\overleftrightarrow{BF}$.



By axiom II-4, since $m$ intersects $\overline{AC}$ at $B$, then it must intersect $\overline{EC}$ or $\overline{AE}$. If it intersected $\overline{EC}$, this must be at point $F$, but then $E * F * C$, which is not possible, by axiom II-3. So, $m$ intersects $\overline{AE}$ at $G$. That is, $A * G * E$. Similarly the line $\overleftrightarrow{CE}$ intersects $\overline{DG}$ at $H$.



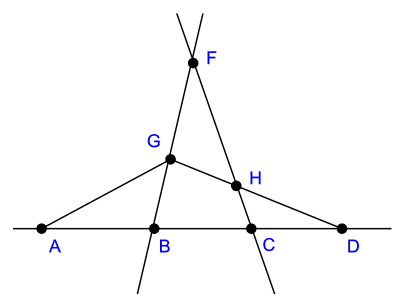

Thus, we have $\overleftrightarrow{EF}$ intersecting $\overline{DG}$ at $H$. By axiom II-4, $\overleftrightarrow{EF}$ must intersect $\overline{AG}$ or $\overline{AD}$. If $\overleftrightarrow{EF}$ intersected $\overline{AG}$, it must do so at $E$, but then $A * E * G$, which contradicts (by axiom II-3) the fact that $A * G * E$. Thus, $\overleftrightarrow{EF}$ intersects $\overline{AD}$, which we know is at point $C$, and $C$ is on $\overline{AD}$.

If we begin with point $B$ rather than $C$ in the argument above, we can likewise prove that $B$ is on $\overline{AD}$. □

**Theorem 11.9.** *Let $A$, $B$, $C$, and $D$ be distinct points on a line $l$. If $B$ is on $\overline{AC}$ and $C$ is on $\overline{AD}$, then $C$ is on $\overline{BD}$ and $B$ is on $\overline{AD}$. (That is, if $A * B * C$ and $A * C * D$, then $B * C * D$ and $A * B * D$)*

Proof: By Theorem 11.2 there must be a point $G$ that is not on $l$. By axiom II-2 we can find a point $F$ such that $B * G * F$.

If the line $\overleftrightarrow{CF}$ intersected $\overline{GB}$, it must do so at $F$, but then $G * F * B$, which contradicts axiom II-3. So, $\overleftrightarrow{CF}$ cannot intersect $\overline{GB}$. We conclude that, since $\overleftrightarrow{CF}$ intersects $\overline{BD}$, then by axiom II-4, it must intersect $\overline{GD}$ at $H$.




Thus, we have $\overleftrightarrow{FH}$ intersecting $\overline{DG}$ at $H$. By axiom II-4, $\overleftrightarrow{FH}$ must intersect $\overline{BG}$ or $\overline{BD}$. If $\overleftrightarrow{FH}$ intersected $\overline{BG}$, it must do so at $F$, but then $B * F * G$, which contradicts (by axiom II-3) the fact that $B * G * F$. Thus, $\overleftrightarrow{FH}$ intersects $\overline{BD}$, which we know is at point $C$, and $C$ is on $\overline{BD}$.

The rest of the proof follows from the previous theorem. □

The next theorem delineates the *line separation* property.

**Theorem 11.10.** *(Line Separation) Let $A$ be a point on a line $l$. Then, $A$ divides the set of all points $X \neq A$ of $l$ into two disjoint classes, with two distinct points being in the same class if and only if $A$ is not between them.*

Proof: Let $B \neq A$ be a point on $l$. Let $S_1$ be the set of all points $C$ on $l$ such that $C = B$ or $A * B * C$ or $A * C * B$. Let $S_2$ be the set of points $D$ on $l$ such that $D * A * B$. By axioms II-1 and II-3, there are no points in both $S_1$ and $S_2$, so these sets are disjoint.

Let $X \neq A$ be a point on $l$. If $X = B$, then $X$ is in $S_1$. Otherwise, by axiom II-3, one of $A * B * X$, $A * X * B$, or $X * A * B$ holds. Equivalently, $X$ is in $S_1$ or $S_2$. Thus, the union of $S_1$ and $S_2$ is equal to the set of points $X \neq A$ on $l$.

Now, we must prove that two distinct points are in the same class if and only if $A$ is not between them. Suppose two points $C$ and $D$ are in $S_1$. If $C = B$, then $D \neq B$ and $A * B * D$ or $A * D * B$, so $A$ is not between $C$ and $D$. If $D = B$, a similar argument shows $A$ is not between $C$ and $D$. If $D = B$, we can use a similar argument to show $A$ is not between $C$ and $D$.

If $C \neq D \neq B$, then we have several options for what points are between other points. If we have $A * B * C$ and $A * D * B$, then by Theorem 11.9, we have $A * D * C$. Likewise, if $A * C * B$ and $A * B * D$, we can show $A * C * D$. If $A * B * C$ and $A * B * D$, could $A$ be between $C$ and $D$? Suppose $C * A * D$. Then, $D * B * A$ and $D * A * C$ imply that $B * A * C$ by the four-point properties. But, this contradicts $A * B * C$, so $A$ is not between $C$ and $D$. One can likewise show that $A$ is not between $C$ and $D$ for all other possible arrangements of $A$,$B$,$C$, and $D$.

Now, we must show that if a point $C$ is in $S_1$ and $D$ is another point such that $A$ is not between $C$ and $D$, then $D$ is also in $S_1$. We know that $A * B * C$ or $A * C * B$. Also, $A * D * C$ or $A * C * D$. Suppose that $A * B * C$ and $A * D * C$. If $B * A * D$, then the four-point properties (with $A * D * C$) imply that $B * A * C$, which contradicts $A * B * C$. So, either $A * B * D$ or $A * D * B$ and $D$ is in $S_1$. Again, we can use the four-point properties to prove that any arrangement of the points yields $D$ in $S_1$.

It will be left as an exercise to show that two distinct points are in $S_2$ if and only if $A$ is not between them. □

This theorem tells us that a point on a line divides, or separates, the line into two *sides*. Thus, when proving results, we are justified in using language that refers to one side or the other on a line.

**Definition 11.3.** *We say two points $B$ and $C$ on a line $l$ are on the* same side *of $A$ on $l$ if $B = C$ or $A$ is not between $B$ and $C$. We say $B$ and $C$ are on* opposite sides *of $A$ on $l$ if they are not on the same side of $A$.*

Given a point $A$ on a line $l$ and and a point $B \neq A$, the ray $\overrightarrow{AB}$ is defined in terms of betweenness properties. Axiom II-2 guarantees the existence of a point $C$ with $C * A * B$. Thus, $C$ is on the opposite side to $B$ from $A$. By the preceding theorem, all points on the ray $\overrightarrow{AC}$ will be opposite to $B$.

**Definition 11.4.** *Two rays on a line $l$ are* opposite *if they share only one point in common.*

The following Corollary follows immediately.

**Corollary 11.11.** *To a given ray $\overrightarrow{AB}$ there is a point $C$ with $\overrightarrow{AC}$ opposite to $\overrightarrow{AB}$.*

The preceding set of theorems dealt with the separation of points on a line. The next definition deals with the separation of points in the plane.

**Definition 11.5.** *Let $l$ be a line and $A$ and $B$ points not on $l$. If $A = B$ or segment $\overline{AB}$ contains no points on $l$ we say that $A$ and $B$ are* on the same side *of $l$. If $A$ is not equal to $B$ and $\overline{AB}$ intersects $l$ we say that $A$ and $B$ are* on opposite sides *of $l$.*

**Theorem 11.12.** *(Plane Separation) For every line $l$ and triple of points $A$, $B$, and $C$ not on $l$ we have:*

1. *If $A$ and $B$ are on the same side of $l$ and $B$ and $C$ are on the same side, then $A$ and $C$ must be on the same side of $l$.*
2. *If $A$ and $B$ are on opposite sides of $l$ and $B$ and $C$ are on opposite sides, then $A$ and $C$ must be on the same side of $l$.*
3. *If $A$ and $B$ are on opposite sides of $l$ and $B$ and $C$ are on the same side, then $A$ and $C$ must be on opposite sides of $l$.*

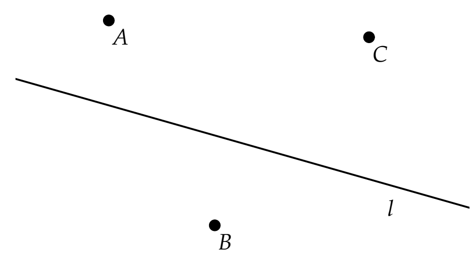



Figure 11.2 Plane Separation

Proof:

1. If $A = B$ or $B = C$ the result is clear. We can assume $A \neq B \neq C$. Suppose $A$ and $C$ were on opposite sides of $l$ (Figure 11.2). Then, line $l$ would intersect $\overline{AC}$ and by axiom II-4 it would have to intersect one of $\overline{AB}$ or $\overline{BC}$, which contradicts the hypothesis.

The proofs of the second and third statements are left as exercises. □

The next theorem guarantees that a line separates the plane into two distinct sides.

**Theorem 11.13.** *Every line $l$ has exactly two sides and these two sides have no points in common.*

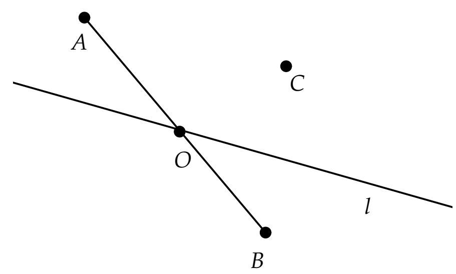



Figure 11.3

Proof: There exists a point $O$ on $l$ and a point $A$ not on $l$ by the incidence axioms (Figure 11.3). Axiom II-2 says that there is a point $B$ with $B * O * A$. Then, by definition, $B$ and $A$ are on opposite sides of $l$ and $l$ has at least two sides. Let $C$ be any other point not on $l$ and not equal to $A$ or $B$. By the Plane Separation Theorem if $A$ and $C$ are on opposite sides, then $C$ and $B$ are on the same side. Likewise, if $B$ and $C$ are on opposite sides, then $C$ and $A$ are on the same side. In any event, $C$ is on one of two sides. If $C$ were on both sides, then by the first part of the Plane Separation Theorem we would have that $A$ and $B$ would be on the same side. This contradicts the fact that they are on opposite sides. □

The next two theorems guarantee that if the endpoints of a segment are not on opposite sides of a line, then the internal points of the segment are on the same side of the line.

**Theorem 11.14.** *Given a line $l$ and two points $A$ and $B$ on the same side of $l$, then all points on segment $\overline{AB}$ are on this same side of $l$.*

Proof: Let $C$ be a point on $\overline{AB}$. If $C$ was on the opposite side to $A$ (or $B$) then segment $\overline{AC}$ would intersect $l$ at some point $P$, with $A * P * C$. We know that $A * C * B$ since $C$ is on $\overline{AB}$. Thus, $A * P * B$, by the four-point properties. But, this contradicts the fact that $A$ and $B$ are on the same side of $l$. □

**Theorem 11.15.** *Given a line $l$ and two points $A$ and $B$ with $A$ on $l$ and $B$ not on $l$, then all points on segment $\overline{AB}$ other than $A$ are on the same side of $l$ as $B$.*

Proof: This is basically a direct consequence of axiom II-3 and is left as an exercise. □

The next two theorems guarantee that we can split a segment or line into two distinct parts.

**Theorem 11.16.** *Given $A * B * C$. Then $\overline{AC} = \overline{AB} \cup \overline{BC}$ and $B$ is the only point common to $\overline{AB}$ and $\overline{BC}$.*

Proof: We will first show that $\overline{AC} \subset \overline{AB} \cup \overline{BC}$. Let $P$ be a point on $\overline{AC}$. (i.e. $A * P * C$) We know that points $A$,$B$,$C$, and $P$ are all on the same line and that there is another point $Q$ not on this line. We also know that line $\overleftrightarrow{PQ}$ exists.

Suppose that $A$ and $B$ were on the same side of $\overleftrightarrow{PQ}$. We know that $A$ and $C$ must be on opposite sides, as $A$ intersects $\overleftrightarrow{PQ}$ at $P$. Thus, $B$ and $C$ must be on opposite sides by the Plane Separation Theorem. Then $\overline{BC}$ must intersect $\overleftrightarrow{PQ}$. But, $\overleftrightarrow{PQ}$ already intersects the line through $\overline{BC}$ at $P$. Thus, since lines can only have a single intersection point, we must have that $\overleftrightarrow{PQ}$ and $\overline{BC}$ intersect at $P$, and $P$ is on $\overline{BC}$.

If $A$ and $B$ are on opposite sides of $\overleftrightarrow{PQ}$, then $P$ is on $\overline{AB}$.

Thus, if $P$ is a point on $\overline{AC}$ then $P$ is on $\overline{AB}$ or on $\overline{BC}$. So, $\overline{AC} \subset \overline{AB} \cup \overline{BC}$.

Now, we will show that $\overline{AB} \cup \overline{BC} \subset \overline{AC}$. If $P$ is on $\overline{AB}$ then $A*P*B$. We know that $A * B * C$. Thus, by the four-point properties, we know that $A * P * C$. Likewise, if $P$ is on $\overline{BC}$ then $C * P * B$. Also, $C * B * A$. Again, using the four-point properties, we get $C * P * A$. Thus, if $P$ is on either $\overline{AB}$ or $\overline{BC}$, we have that $P$ is on $\overline{AC}$. So, $\overline{AB} \cup \overline{BC} \subset \overline{AC}$. We have already shown that $\overline{AC} \subset \overline{AB} \cup \overline{BC}$, so we conclude that $\overline{AC} \subset \overline{AB} = \overline{BC}$.

Finally, why will $B$ be the only common point to $\overline{AB}$ and $\overline{BC}$? Suppose $R$ is another point common to these two segments with $R \neq B$. We know there is another point $S$ not collinear with $A$, $B$, $C$, and $R$. Consider the line $\overleftrightarrow{RS}$. Since $R$ is on $\overline{AB}$ and on $\overline{BC}$ and $R \neq B$ then $A$ and $B$ are on opposite sides of $\overleftrightarrow{RS}$ and so are $B$ and $C$. Thus,

$A$ and $C$ must be on the same side of $\overleftrightarrow{RS}$ by the Plane Separation Theorem. So, $\overline{AC}$ must not intersect $\overleftrightarrow{RS}$. However, we just proved that $\overline{AC} = \overline{AB} \cup \overline{BC}$, and both $\overline{AB}$ and $\overline{BC}$ intersect $\overleftrightarrow{RS}$. This leads to a contradiction and thus $B$ is the only common point to $\overline{AB}$ and $\overline{BC}$. □

**Theorem 11.17.** *Given $A * B * C$. Then $\overrightarrow{AB} = \overrightarrow{AC}$ and $B$ is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.*

Proof: First, we will show that $\overrightarrow{AB} \subset \overrightarrow{AC}$. Let $P$ be on $\overrightarrow{AB}$. If $P$ is on $\overline{AC}$ then it is on $\overrightarrow{AC}$. Otherwise, by axiom II-3 either $P * A * C$ or $A * C * P$. If $A * C * P$ then $P$ is on $\overrightarrow{AC}$. If $P * A * C$, then $C * A * P$. Since $C * B * A$, by the four-point properties we have $B * A * P$, which contradicts $P$ being on ray $\overrightarrow{AB}$. Thus, $\overrightarrow{AB} \subset \overrightarrow{AC}$.

Next we show that $\overrightarrow{AC} \subset \overrightarrow{AB}$. Let $P$ be on $\overrightarrow{AC}$. Then, $A * P * C$ or $A * C * P$.

Suppose $A * P * C$. Then $P$ is on segment $\overline{AC}$. By the previous theorem $\overline{AC} = \overline{AB} \cup \overline{BC}$, so either $P$ is either on $\overline{AB}$ or on $\overline{BC}$. If $P$ is on $\overline{AB}$ then clearly $P$ is on $\overrightarrow{AB}$. If $P$ is on $\overline{BC}$ then $C * P * B$. We are given that $C * B * A$. Thus, by the four-point properties we have $P * B * A$ (or $A * B * P$) and $P$ is again on $\overrightarrow{AB}$.

Now, suppose that $A * C * P$. We are given that $A * B * C$. By the four-point properties we have that $A * B * P$ and $P$ is on $\overrightarrow{AB}$.

Thus, $\overrightarrow{AC} \subset \overrightarrow{AB}$, and so $\overrightarrow{AC} = \overrightarrow{AB}$ The proof that $B$ is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is left as an exercise. □

This finishes our development of the notion of order for points, segments, and lines. In the next section we expand this notion of order to include angles.

**Exercise 11.2.1.** *Prove Theorem 11.2*

**Exercise 11.2.2.** *Prove Theorem 11.3*

**Exercise 11.2.3.** *Prove Theorem 11.4*

**Exercise 11.2.4.** *Prove Theorem 11.5*

**Exercise 11.2.5.** *In the proof of Theorem 11.10, show that two distinct points $C$ and $D$ are in $S_2$ if and only if $A$ is not between them (refer to the Theorem for definitions of $S_2$ and $A$). [Hint: Consider the relation of $B$ to $D$ and $C$.]*

**Exercise 11.2.6.** *Prove the second part of the Plane Separation Theorem.*

**Exercise 11.2.7.** *Prove the third part of the Plane Separation Theorem.*

**Exercise 11.2.8.** *Prove Theorem 11.15.*

**Exercise 11.2.9.** *Finish the proof of Theorem 11.17. That is, show that if $A * B * C$, then $B$ is the only point common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$.*

**Exercise 11.2.10.** *Prove that if $A * B * C$ then segment $\overline{AB}$ is contained in segment $\overline{AC}$.*

**Exercise 11.2.11.** *Given a line $l$, a point $A$ on $l$, and a point $B$ not on $l$, show that every point $C \neq A$ on $\overrightarrow{AB}$ is on the same side of $l$ as $B$. [Hint: Show by contradiction.]*

**Exercise 11.2.12.** *Prove that a line must have an infinite number of points. [Hint: Start with two points on a line and repeatedly use Theorem 11.7 and Theorem 11.16.]*

## 11.3 PROJECT 19 - ANGLES AND RAY BETWEENNESS

Many of the projects we have covered so far involve computer exploration of geometric topics. In this project we will explore a more abstract idea —the betweenness property defined by rays and angles. In this project our exploratory "canvas" will be the canvas of our minds. Feel free to draw diagrams for each new idea, but be careful to make your arguments based *solely* on the theorems and axioms of preceding sections.

To effectively explore the properties of angles, we need a good definition. The following definition follows the development used by Euclid and precludes an angle of 0° or 180°.

**Definition 11.6.** *A pair of rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, not lying on the same line, is called an* angle *and is denoted by $\angle CAB$. The two rays are called the* sides *of the angle and the common point $A$ is called the* vertex *of the angle.*

We have looked at several notions of "side" in this chapter. There was the notion of two sides on a line determined by a point on a line (Line Separation). There was the notion of a line dividing the plane into two sides (Plane Separation). Now, we have sides of an angle. Usually, we will be able to infer which of these different uses of "side" is applicable by context. But, if there is the danger of confusion, we will refer to the sides of an angle as "angle sides."

> **Definition 11.7.** *Given an angle $\angle CAB$ a point $D$ is in the* interior *of the angle if $D$ is on the same side of $\overleftrightarrow{AB}$ as $C$ and if $D$ is on the same side of $\overleftrightarrow{AC}$ as $B$.*

The definition implies that the interior of an angle is equal to the intersection of two sides of lines.

**Exercise 11.3.1.** *Your first task is to use the axioms, definitions, and theorems of the preceding section to show the following result.*

> **Theorem 11.18.** *Given an angle $\angle CAB$, if $D$ is a point lying on line $\overleftrightarrow{BC}$, then $D$ is in the interior of the angle if and only if $B * D * C$. (see Figure 11.4)*

*[Hint: This requires two proofs. For the first, assume $D$ is in the interior and $D$ is* not *between $B$ and $C$. Can you reach a contradiction? For the second, assume $B * D * C$ and $D$ is* not *in the interior of the angle. Then, either $\overline{DB}$ or $\overline{DC}$ intersect a side. Can you find a contradiction?]*

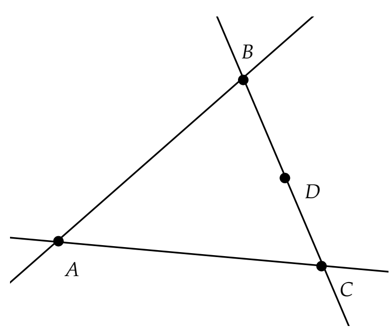



Figure 11.4

Note that this theorem implies that if $E$ and $F$ are two points on the two angle sides of $\angle CAB$, with $E \neq A$ and $F \neq A$, then all points on the segment $\overline{EF}$ are in the interior of the angle.

It is also an immediate consequence of Theorem 11.14 that if two points $D$ and $E$ are in the interior of an angle then all points on $\overline{DE}$ are interior to that angle. (Verify this)

Additionally, from Theorem 11.15 we have that if $D$ is interior to an angle and $B$ is on one of the two angle sides, and not the vertex, then all points on $\overline{BD}$ other than $B$ will be interior to the angle. (Verify this)

Thus, we have the following:

**Theorem 11.19.** *Given $\angle CAB$ let $E$ and $F$ two points with*

- *$E$ and $F$ on different angle sides, or*
- *$E$ on an angle side and $F$ interior, or*
- *$E$ and $F$ interior to the angle.*

*then all points on $\overline{EF}$ are interior to the angle, (except for endpoints on the angle sides in the first two cases.)*

**Exercise 11.3.2.** *Your next task is to prove the following result on rays within an angle.*

**Theorem 11.20.** *If $D$ is a point in the interior of $\angle CAB$ then*

- *All other points on ray $\overrightarrow{AD}$ except $A$ are also in the interior.*
- *No point on the opposite ray to $\overrightarrow{AD}$ is in the interior.*
- *If $C * A * E$ then $B$ is in the interior of $\angle DAE$.*

*[Hint: The situation is illustrated in Figure 11.5. For the first part of the Theorem, use Exercise 11.2.11. For the second part, use contradiction. For the third part, show that $\overline{EB}$ does not intersect $\overrightarrow{AD}$ and does not intersect the ray opposite to $\overrightarrow{AD}$. Then, use what you know about $D$ to finish the proof.]*

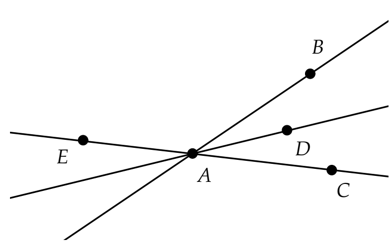



Figure 11.5

We now define a betweenness property for rays.

**Definition 11.8.** *A ray* $\overrightarrow{AD}$ *is* between *rays* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *if* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are not opposite rays and* $D$ *is interior to* $\angle CAB$*.*

Note that the previous theorem guarantees that this definition does not depend on the choice of $D$ on $\overrightarrow{AD}$.

Review the following theorem and its proof carefully.

**Theorem 11.21.** *(Crossbar Theorem) If* $\overrightarrow{AD}$ *is between* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *then* $\overrightarrow{AD}$ *intersects segment* $\overline{BC}$*.*

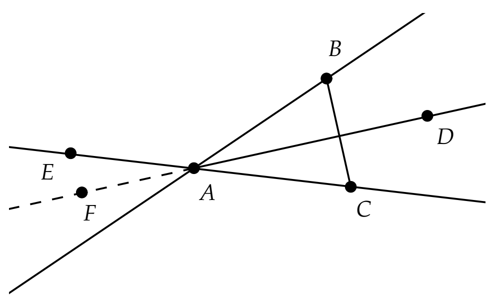



Figure 11.6

Proof: This theorem derives its name from the suggestive shape of Figure 11.6. Assume that $\overrightarrow{AD}$ does not intersect segment $\overline{BC}$. Let $\overrightarrow{AF}$ be the ray opposite to $\overrightarrow{AD}$. If $\overrightarrow{AF}$ intersects $\overline{BC}$ at some point $P$ then $B * P * C$ and by Theorem 11.18 we have that $P$ is interior to $\angle CAB$. But this contradicts Theorem 11.20, part ii.

Since $\overleftrightarrow{AD} = \overrightarrow{AD} \cup \overrightarrow{AF}$ we have that $\overleftrightarrow{AD}$ does not intersect $\overline{BC}$. Thus, $B$ and $C$ are on the same side of $\overleftrightarrow{AD}$.

Now, let $E$ be a point on $\overleftrightarrow{AC}$ with $E * A * C$. ($E$ exists by axiom II-2). $C$ and $E$ are then on opposite sides of $\overleftrightarrow{AD}$.

Now, since $B$ and $C$ are on the same side of $\overleftrightarrow{AD}$ and $C$ and $E$ are on opposite sides we must have (by the Plane Separation Theorem) that $B$ and $E$ are on opposite sides. However, $B$ is in the interior of $\angle DAE$ by part iii of the previous theorem. So, $B$ and $E$ must be on the same side of $\overleftrightarrow{AD}$.

Thus, the assumption that $\overrightarrow{AD}$ does not intersect segment $\overline{BC}$ leads to a contradiction, so $\overrightarrow{AD}$ must intersect $\overline{BC}$. □

Note that since $\overrightarrow{AD}$ must intersect segment $\overline{BC}$ at some point, we can assume that $D$ is that point of intersection. This allows us to say that if $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then $B * D * C$. The converse is also true.

**Exercise 11.3.3.** *Prove the following converse to the Crossbar Theorem.*

> **Theorem 11.22.** *Given $B * D * C$ there exists a point $A$ not on $\overleftrightarrow{BC}$ such that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.*

*[Hint: Theorem 11.2 guarantees the existence of a point $A$ not on $\overleftrightarrow{BC}$. Consider $\angle BAC$ and use Theorem 11.18]*

We have shown that a ray is *between* two other rays iff a point is between two other points. This says that points and rays are essentially inter-changeable when it comes to the property of betweenness. This property of inter-changeability is called *duality*. One amazing implication of this is that any result on point betweenness will automatically generate a corresponding result for ray betweenness, without the need for a proof!

For example,

> **Theorem 11.23.** *Suppose that we have rays $\overrightarrow{EA}$, $\overrightarrow{EB}$, $\overrightarrow{EC}$, and $\overrightarrow{ED}$. If $\overrightarrow{EB}$ is between $\overrightarrow{EA}$ and $\overrightarrow{EC}$, and $\overrightarrow{EC}$ is between $\overrightarrow{EB}$ and $\overrightarrow{ED}$, then $\overrightarrow{EB}$ is between $\overrightarrow{EA}$ and $\overrightarrow{ED}$ and $\overrightarrow{EC}$ is between $\overrightarrow{EA}$ and $\overrightarrow{ED}$.*

**Exercise 11.3.4.** *Find the Theorem in the last section for which this is the dual.*

### *Project Report*

In this project we have developed the notion of angle and the betweenness properties of of rays. In your project report provide clear and complete solutions to the exercises. In your conclusion discuss briefly the significance of the axioms of order. Why are the results of this project, and the preceding section, so important in a solid development of geometry?

## 11.4 TRIANGLES AND BETWEENNESS

We now take a brief excursion into how betweenness and separation is related to triangles.

**Definition 11.9.** *Given three non-collinear points $A$, $B$, and $C$, the* triangle $\Delta ABC$ *is the set of points on the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. These segments are called the* sides *of the triangle.*

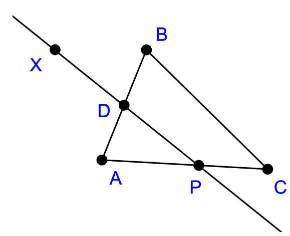
**Definition 11.10.** *The* interior *of triangle $\Delta ABC$ is the intersection of the interior of its angles $\angle CAB$, $\angle ABC$, $\angle BCA$. A point is in the* exterior *of the triangle if it is not in the interior and is not on any side.*

The following two theorems gives "Pasch-like" results on the intersection properties of rays with triangles.

**Theorem 11.24.** *Given $\Delta ABC$, if a ray $r$ starts at an exterior point of the triangle and intersects side $\overline{AB}$ at a single point $D$ with $A * D * B$ then then this ray must also intersect one of the other two sides of the triangle.*

Proof:

Let the ray $r$ be given as $\overrightarrow{XD}$, where $X$ is the exterior point. By Pasch's axiom (axiom II-4), we have that *line* $\overleftrightarrow{XD}$ will intersect either $\overline{AC}$ or $\overline{BC}$ at some point $P$. We need to show $P$ is on $\overrightarrow{XD}$.




By axiom II-3 either $X * D * P$, $X * P * D$ or $P * X * D$. In either of the first two cases $P$ is on $\overrightarrow{XD}$.

Suppose $P * X * D$. If $P = C$, then by Theorem 11.18, $D$ is interior to the angle $\angle BCA$ and by Theorem 11.20, $X$ is also interior to this angle. But, this contradicts the fact that $X$ is exterior.

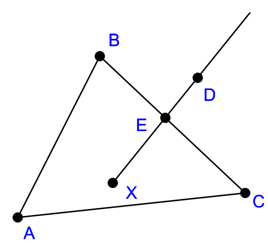
Now, suppose $P * X * D$ and $P \neq C$. Pasch's axiom says that $\overleftrightarrow{XD}$ only intersects one of $\overline{AC}$ or $\overline{BC}$. We assume it intersects $\overline{AC}$, where

$P$ is on $\overline{AC}$. Since $D$ is interior to $\angle ACB$, by Theorem 11.19 we know that $X$ is interior to $\angle ACB$. Also, since $\angle CAB$ is the same as $\angle PAD$ ($\overrightarrow{AB} = \overrightarrow{AD}$ and $\overrightarrow{AC} = \overrightarrow{AP}$) we have by Theorem 11.18 that $X$ is interior to $\angle CAB$. Lastly, we know that $A * P * C$. By Theorem 11.18 we have that $P$ is interior to $\angle ABC$ and thus, by Theorem 11.19, $X$ is interior to $\angle ABC$. But, if $X$ is interior to all three angles it is interior to the triangle, which is a contradiction. □

**Theorem 11.25.** *Given $\Delta ABC$, if a ray $r$ starts at an interior point of the triangle then it must intersect one of the sides. If it does not pass through a vertex it intersects only one side.*

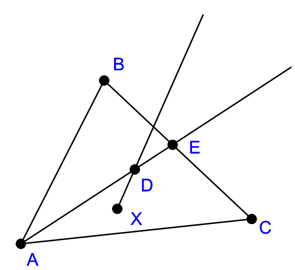
Proof: Let ray $r$ be given as $\overrightarrow{XD}$ where $X$ is the interior point inside the triangle and $D$ is another point on the ray.

If $D$ is exterior to the triangle then $X$ and $D$ are on opposite sides of one of the sides, and thus $\overline{XD}$ must intersect this side. Since segment $\overline{XD}$ is on $\overrightarrow{XD}$ then the ray intersects a side.




If $D$ is on one of the sides of the triangle, clearly the results holds.

If $D$ is interior to the triangle, then $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. By the Crossbar Theorem $\overrightarrow{AD}$ intersects $\overline{BC}$ at some point $E$, and since $D$ is interior to the triangle, we must have that $A * D * E$. ($A$ and $D$ are on the same side of $\overline{BC}$.)




Now, either $X$ is on the same side of $\overleftrightarrow{AE}$ as $C$ or is on the opposite side, which is the side $B$ is on. Thus, $X$ is exterior to one of the two triangles $ACE$ and $AEB$. Let's say it is exterior to $AEB$. Then, by Theorem 11.24 we have that $\overrightarrow{XD}$ intersects side $\overline{AB}$ or $\overline{BE}$. A simple argument shows that $\overline{BE}$ is contained in $\overline{BC}$. So, in all cases $\overrightarrow{XD}$ intersects a side of $ABC$. □

We end this brief discussion of triangles with a nice result that we will need later when discussing acute and obtuse angles.

**Theorem 11.26.** *Let $\overleftrightarrow{AA'}$ be a line with $A * O * A'$. Let $B \neq C$ be two points on the same side of $\overleftrightarrow{AA'}$. If $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, then $\overrightarrow{OC}$ is between $\overrightarrow{OB}$ and $\overrightarrow{OA'}$. (Figure 11.7)*

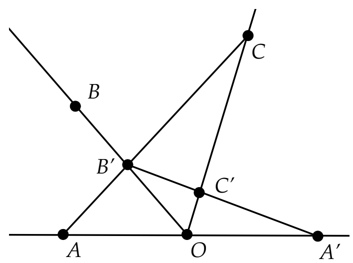



Figure 11.7

Proof: By the Crossbar Theorem, we know that $\overrightarrow{OB}$ will intersect $\overline{AC}$ at some point $B'$. Then, since $\overrightarrow{OB} = \overrightarrow{OB'}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, we have $A * B' * C$ by the duality of betweenness for points and angles. Consider $\Delta AA'B'$. We have that $\overrightarrow{OC}$ intersects side $\overline{AA'}$. By Theorem 11.24, this ray must then intersect one of $\overline{AB'}$ or $\overline{B'A'}$. But, if it intersects $\overline{AB'}$ then this intersection point must be $C$ since $C$ is already on $\overleftrightarrow{AB'}$ and the lines $\overleftrightarrow{OC}$ and $\overleftrightarrow{AB}$ can only intersect once. But, if the intersection point is $C$ then $A * C * B'$, which contradicts the fact that $A * B' * C$. So, $\overrightarrow{OC}$ must intersect $\overline{B'A'}$ at some point $C'$. Then, $B' * C' * A'$ and $\overrightarrow{OC'}$ is between $\overrightarrow{OB'} = \overrightarrow{OB}$ and $\overrightarrow{OA'}$. This finishes the proof. □

**Exercise 11.4.1.** *Prove that a line cannot be contained in the interior of a triangle.*

**Exercise 11.4.2.** *Given $\Delta ABC$, if a ray $r$ starts at vertex $A$ of the triangle and passes through an interior point, show that it must intersect $\overline{BC}$.*

**Exercise 11.4.3.** *Given $\Delta ABC$, if a ray $r$ starts at an exterior point of the triangle and intersects side $\overline{AB}$ at a single point $D$ with $A * D * B$, show that this ray contains an interior point of the triangle.*

**Exercise 11.4.4.** *Show that the interior of a triangle is a nonempty set of points.*

## 11.5 CONGRUENCE GEOMETRY

Just as the notions of incidence and betweenness were left undefined in the preceding sections, the notion of congruence for segments and angles will be left as undefined in this section. Our intuitive idea of congruence tells us that two segments are the same if one can be exactly overlayed on top of the other. This intuitive idea assumes the existence of functions that *transform* segments to other segments. This transformational geometry presupposes an existing set of transformations that would itself need an axiomatic basis.

By leaving the idea of congruence undefined for segments and angles we allow for interpretations via different models. As long as the notion of congruence in a particular model satisfies the following axioms, we can apply subsequent theorems derived from those axioms to this model.

What the congruence axioms give us is a basis for comparing segments and rays in a similar fashion to how we compare numbers. Equality for numbers is a property that is *reflexive* (every number equals itself), *symmetric* (if $a = b$ then $b = a$), and *transitive* ( if $a = b$ and $b = c$, then $a = c$). These three properties are critical to how we construct an arithmetic system.

Additionally, the axioms will give us a way to "add" and "subtract" segments and angles, again providing algebraic properties to the geometry.

We will need the following definition of triangle congruence.

> **Definition 11.11.** *Two triangles are* congruent *if there is some way to match vertices of one to the other such that corresponding sides are congruent and corresponding angles are congruent.*

If $\Delta ABC$ is congruent to $\Delta A'B'C'$ we shall use the notation $\Delta ABC \cong \Delta A'B'C'$. Thus, $\Delta ABC \cong \Delta A'B'C'$ if and only if

$$\overline{AB} \cong \overline{A'B'}, \overline{AC} \cong \overline{A'C'}, \overline{BC} \cong \overline{B'C'}, \angle A \cong \angle A', \angle B \cong \angle B', and \angle C \cong \angle C'.$$

We use the symbol "≅" to represent the undefined notion of congruence for segments and angles.

Here are the six axioms of congruence:

- **III-1 If $A$ and $B$ are distinct points and $A'$ is any other point, then for each ray $r$ from $A'$ there is a unique point $B'$ on $r$ such that $B' \neq A'$ and $\overline{AB} \cong \overline{A'B'}$.**

- **III-2 If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$ then $\overline{CD} \cong \overline{EF}$. Also, every segment is congruent to itself.**

- **III-3 If $A * B * C$ , $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$ , and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.**

- **III-4 Given $\angle BAC$ and given any ray $\overrightarrow{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on a given side of $\overleftrightarrow{A'B'}$ such that $\angle BAC \cong \angle B'A'C'$.**

- **III-5 If $\angle BAC \cong \angle B'A'C'$ and $\angle BAC \cong \angle B''A''C''$ then $\angle B'A'C' \cong \angle B''A''C''$. Also, every angle is congruent to itself.**

- **III-6 Given two triangles $\Delta ABC$ and $\Delta A'B'C'$ if $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$ then the two triangles are congruent.**

Let's analyze these axioms a bit. First of all, axiom III-1 basically covers our intuitive idea of segments being congruent via transformation. We can think of $A$ being moved to $A'$ and then $B'$ is where $B$ would move under the transformation. Axiom III-1 implies both Proposition 2 and Proposition 3 of Book I of *Elements*.

Another simple implication of this axiom is that if $A, B, C$ are points on a line with $A * B * C$ ($B$ is on $\overline{AC}$), and $B \neq C$, then $\overline{AB}$ cannot be congruent to $\overline{AC}$.

Axiom III-2 covers a property that is like transitivity. III-2 also says that congruence is a reflexive property. This is what Euclid would call a "common notion". We note here that this axiom implies the symmetry of congruence, since if $\overline{AB} \cong \overline{A'B'}$ and $\overline{AB} \cong \overline{AB}$ then by III-2 we have that $\overline{A'B'} \cong \overline{AB}$. Also, once we have symmetry we have full transitivity, since if $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{CD} \cong \overline{AB}$ and $\overline{CD} \cong \overline{EF}$, so $\overline{AB} \cong \overline{EF}$.

Axiom III-3 covers the intuitive idea that congruent segments that are joined to other congruent segments make new segments that are again congruent. This axiom can also be interpreted in terms of "adding" congruent segments. It is sometimes called the "Segment Addition" axiom.

Axioms III-4 and III-5 guarantee that angles can be "laid off" onto other rays and that angle congruence is transitive and reflexive (and thus symmetric), just as was the case with segment congruence. Axiom III-4 is also the same as Proposition 23 of Book I of *Elements*.

It turns out that there is no need for an axiom on joining angles to angles (i.e. an addition axiom for angles comparable to Axiom III-3). This fact will be proven below.

Axiom III-6 is the Side-Angle-Side (SAS) congruence result that is familiar from classical Euclidean geometry. It is Proposition 4 of Book I of *Elements*. Euclid proved this result by an argument that was based on "moving" or transforming points. As was mentioned above, this approach can work if the notion of transformation is axiomatized. However, SAS cannot be proven from the rest of Hilbert's axioms. Hilbert, in his classic work "Foundations of Geometry" [13] constructs a model of geometry based on all of his axioms, *except* the SAS axiom. He shows that this model is consistent, but the SAS result does not universally hold. This shows that SAS is *independent* from the other axioms, and thus must be stated as an axiom, if it is to be used.

We note that the congruence axioms are valid in all of our models of Hyperbolic geometry, and are also valid in Elliptic geometry - with a modified set of betweenness axioms used for definitions of segments, angles and separation. We will review this modified set of congruence axioms for Elliptic geometry in section 14.4

### 11.5.1 Triangle and Angle Congruence Results

Using the congruence axioms, we derive some basic facts about triangle and angle congruence.

**Definition 11.12.** *$\Delta ABC$ is called* isosceles *if it has two congruent sides, i.e. $\overline{AB} \cong \overline{AC}$. Segment $\overline{BC}$ is called the* base *of the triangle, and the angles at $B$ and $C$ are called* base angles.

**Theorem 11.27.** *If $\Delta ABC$ is isosceles with $\overline{AB} \cong \overline{AC}$ then $\angle CBA \cong \angle ACB$. That is, in an isosceles triangle, the base angles are congruent.*

The proof of this result, which is also the first part of Proposition 5 of Book I of *Euclid's Elements*, is a straight-forward application of SAS to triangles $\Delta ABC$ and $\Delta ACB$. The second part of the fifth proposition essentially deals with *supplementary angles*.

**Definition 11.13.** *Two angles that have a vertex and side in common and whose separate sides form a line are called* supplementary *angles.*

It is clear that given any angle at least one supplementary angle always exists. For, given $\angle ABC$ the opposite ray to side $\overrightarrow{BC}$ will exist on the line through $B$ and $C$. Let $D$ be a point on this ray. Then, $\angle ABD$ will be supplementary to $\angle ABC$.

The following theorem can be used to prove the second part of Euclid's Proposition 5.

**Theorem 11.28.** *Supplementary angles of congruent angles are congruent.*

Proof: Let $\angle ABC \cong \angle DEF$. Let $\angle ABG$ and $\angle DEH$ be supplementary to $\angle ABC$ and $\angle DEF$.

Since points $A$, $C$ and $G$ are arbitrarily given, we can assume by axiom III-1 that points $D$, $F$ and $H$ are chosen such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{BG} \cong \overline{EH}$. By SAS we then have that $\Delta ABC \cong \Delta DEF$ and thus $AC \cong DF$ and $\angle ACB \cong \angle DFE$. (Figure 11.8)

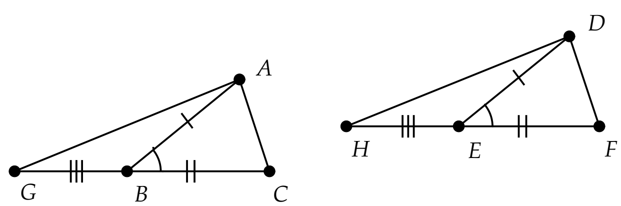



Figure 11.8

Now, it is clear that $G * B * C$, for if this were not the case, then either $B * G * C$, which implies that $G$ is on $\overrightarrow{BC}$, or $G * C * B$, which implies that $C$ is on $\overrightarrow{BG}$. In either case we get a contradiction to the definition of supplementary angles. The addition axiom (III-3) then says that $\overline{CG} \cong \overline{FH}$.

SAS then gives that $\Delta ACG \cong \Delta DFH$. Then, $\angle CGA \cong \angle FHD$

and $\overline{AG} \cong \overline{DH}$. Since $\overline{GB} \cong \overline{HE}$ we have again by SAS that $\Delta AGB \cong \Delta DHE$. Thus, the two supplementary angles $\angle ABG$ and $\angle DEH$ are congruent. $\square$ We note here that the definition of supplementary angles, plus the preceding theorem, can serve as a replacement for Euclid's Proposition 13. This proposition deals with two distinct lines $l$ and $m$ that cross at a point $P$. Proposition 13 states that the angles made by a ray from $P$ on one of the lines will create two angles that add to two right angles. The obvious problem with this statement is that Euclid never defined what is meant by "two right angles." Our definition of supplementary angles solves this problem, and Theorem 11.28 can be used in place of Euclid's Proposition 13 for proofs of basic results. For example, it can be used to show that vertical angles are congruent, which is is Proposition 15 of Book I of *Elements*.

**Definition 11.14.** *Two angles with a common vertex and whose sides form two lines are called* vertical *angles.*

**Theorem 11.29.** *Vertical angles are congruent to each other.*

Proof: This is a direct result from the previous theorem and will be left as an exercise. $\square$

Supplementary angles also provide a simple definition for *right angles*.

**Definition 11.15.** *An angle that is congruent to one of its supplementary angles is called a* right *angle.*

We finish this section with a few basic results on triangle congruence.

**Theorem 11.30.** *(ASA for triangles) Given two triangles $\Delta ABC$ and $\Delta DEF$ with $\angle BAC \cong \angle EDF$ and $\angle ACB \cong \angle DFE$ and $\overline{AC} \cong \overline{DF}$, then the two triangles are congruent (Figure 11.9).*

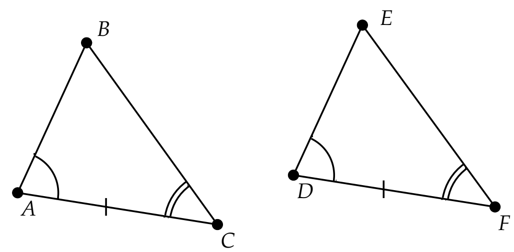



Figure 11.9 ASA

Proof: By axiom III-1 there is a unique point $G$ on $\overrightarrow{DE}$ with $\overline{AB} \cong \overline{DG}$. By SAS $\Delta ABC \cong \Delta DGF$. Thus, $\angle ACB \cong \angle DFG$. By transitivity of angles $\angle DFE \cong \angle DFG$. By axiom III-4 this implies that $E$ and $G$ are both on the same ray $\overrightarrow{EF}$. If $E$ and $G$ were not the same point then the lines $\overleftrightarrow{EF}$ and $\overleftrightarrow{DE}$ would intersect in more than one point, which is impossible. Thus, $\Delta ABC \cong \Delta DEF$. $\square$

Note that this is part of Proposition 26 in Book I of *Elements*. This theorem can be used to show the next theorem, which is the converse of Theorem 11.27, and also Proposition 6 of Book I of *Elements*.

**Theorem 11.31.** *If in $\Delta ABC$ we have $\angle ABC \cong \angle ACB$ then $\overline{AB} \cong \overline{AC}$ and the triangle is isosceles.*

Proof: Exercise. $\square$

### 11.5.2 Segment and Angle Ordering

The congruence axioms not only provide basic notions of equality for segments and angles, they also provide the basis for the *ordering* of these quantities.

To define an ordering for segments we will need the following two theorems. The first is the segment subtraction theorem. (Compare this to axiom III-3 above) The second guarantees that betweenness properties are preserved under segment congruence.

**Theorem 11.32.** *If $A{*}B{*}C$, $D{*}E{*}F$, $\overline{AB} \cong \overline{DE}$, and $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$ (Figure 11.10).*

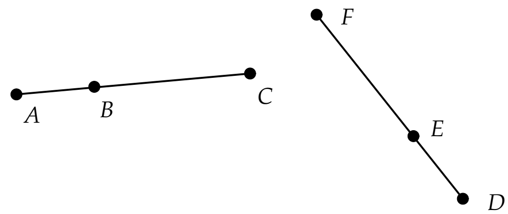



Figure 11.10

Proof: Assume that $\overline{BC}$ is not congruent to $\overline{EF}$. We know from axiom III-1 that there is a point $G$ on $\overrightarrow{EF}$ such that $\overline{BC} \cong \overline{EG}$. Also, $G \neq F$ because if these two points were equal, then $\overline{BC} \cong \overline{EG}$ and $\overline{EG} \cong \overline{EF}$ would imply, by transitivity, that $\overline{BC} \cong \overline{EF}$.

Now, $\overline{AB} \cong \overline{DE}$. Thus, by axiom III-3 we have that $\overline{AC} \cong \overline{DG}$. By transitivity $\overline{DF} \cong \overline{DG}$. But if $F \neq G$ then $\overline{DF} \neq \overline{DG}$ (see the note on axiom III-1 above). □

**Theorem 11.33.** *Given $\overline{AC} \cong \overline{DF}$ then for any point $B$ between $A$ and $C$ there is a unique point $E$ between $D$ and $F$ such that $\overline{AB} \cong \overline{DE}$.*

Proof: By axiom III-1 there is a unique point $E$ on $\overrightarrow{DF}$ such that $\overline{AB} \cong \overline{DE}$. Now, if $E = F$ then from $\overline{AC} \cong \overline{DF}$ and $\overline{AB} \cong \overline{DF}$ we would have $\overline{AC} \cong \overline{AB}$, contradicting axiom III-1, as $B \neq C$.

Suppose $E$ is not between $D$ and $F$, i.e. $D * F * E$. On the opposite ray to $\overrightarrow{CA}$ there is a unique point $G$ with $\overline{FE} \cong \overline{CG}$, by axiom III-1. By the addition axiom we have that $\overline{DE} \cong \overline{AG}$. We already have that $\overline{AB} \cong \overline{DE}$. Thus, $\overline{AB} \cong \overline{AG}$. Now, $A * B * C$, and since $G$ is on the opposite ray to $\overrightarrow{CA}$, then $A*C*G$. By the four-point properties we have that $A*B*G$, and thus $\overline{AB}$ cannot be congruent to $\overline{AG}$ (since $B \neq G$). This contradicts $\overline{AB} \cong \overline{AG}$.

Thus, the only possibility left is that $E$ is between $D$ and $F$. □

We can now define an ordering on segments.

**Definition 11.16.** *We use the notation* $\overline{AB} < \overline{CD}$ *(equivalently* $\overline{CD} > \overline{AB}$*) to mean that there is a point* $E$ *between* $C$ *and* $D$ *such that* $\overline{AB} \cong \overline{CE}$*.*

**Theorem 11.34.** *(Segment Order)*

*(i) One and only one of the following holds:* $\overline{AB} < \overline{CD}$*,* $\overline{AB} \cong \overline{CD}$*, or* $\overline{AB} > \overline{CD}$*.*

*(ii) If* $\overline{AB} < \overline{CD}$ *and* $\overline{CD} \cong \overline{EF}$*, then* $\overline{AB} < \overline{EF}$*.*

*(iii) If* $\overline{AB} > \overline{CD}$ *and* $\overline{CD} \cong \overline{EF}$*, then* $\overline{AB} > \overline{EF}$*.*

*(iv) If* $\overline{AB} < \overline{CD}$ *and* $\overline{CD} < \overline{EF}$*, then* $\overline{AB} < \overline{EF}$*.*

Proof: For the first statement, suppose $\overline{AB} < \overline{CD}$ and $\overline{AB} \cong \overline{CD}$. Then, there is a point $E$ between $C$ and $D$ such that $\overline{AB} \cong \overline{CE}$. By transitivity, $\overline{CD} \cong \overline{CE}$ with $E \neq C$. This contradicts axiom III-1. A similar argument shows that $\overline{AB} > \overline{CD}$ and $\overline{AB} \cong \overline{CD}$ is not possible.

There is only one case left for statement 1 of the theorem, that of $\overline{AB} < \overline{CD}$ and $\overline{AB} > \overline{CD}$. If this is the case then there is a point $E$ between $C$ and $D$ such that $\overline{AB} \cong \overline{CE}$ and also there is a point $F$ between $A$ and $B$ such that $\overline{AF} \cong \overline{CD}$. Now, on the ray opposite $\overrightarrow{BA}$ there is a unique point $G$ with $\overline{BG} \cong \overline{ED}$, by axiom III-1. We also note that $F$ and $G$ are on opposite sides of $B$ and thus cannot be equal. By the addition axiom (III-3) we have that $\overline{AG} \cong \overline{CD}$. By transitivity $\overline{AG} \cong \overline{AF}$, with both on ray $\overrightarrow{AB}$. But, this contradicts axiom III-1 as if $\overline{AG} \cong \overline{AF}$ then $G = F$.

For the second statement of the theorem, if $\overline{AB} < \overline{CD}$ then there is a point $G$ between between $C$ and $D$ such that $\overline{AB} \cong \overline{CG}$. Also, by the previous theorem there is a unique point $H$ between $E$ and $F$ such that $\overline{CG} \cong \overline{EH}$. By transitivity $\overline{AB} \cong \overline{EH}$ and thus by definition $\overline{AB} < \overline{EF}$.

A similar argument proves statement 3 of the theorem.

For the fourth statement, we know there is a point $H$ between between $E$ and $F$ such that $\overline{CD} \cong \overline{EH}$. Thus, $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EH}$. By statement 2 of the theorem $\overline{AB} < \overline{EH}$. Thus, there is a point $K$ between $E$ and $H$ with $\overline{AB} \cong \overline{EK}$. So, $E * K * H$ and $E * H * F$. By the four-point properties we have $E * K * F$. By definition, $\overline{AB} < \overline{EF}$. □

In preceding sections, we discussed the dual nature of segments and angles. Since we have just shown that segments can be ordered, it is not surprising that an order can be defined for angles. We will explore angle order in detail in the next project. In preparation for that work, we will review the basic definitions and properties of angle order.

**Definition 11.17.** *An angle $\angle ABC$ is said to be* less than *an angle $\angle DEF$ (denoted $\angle ABC < \angle DEF$) if there exists a point $G$ interior to $\angle DEF$ with $\angle ABC \cong \angle DEG$. In this case, we also say that $\angle DEF$ is* greater than *$\angle ABC$ (Figure 11.11).*

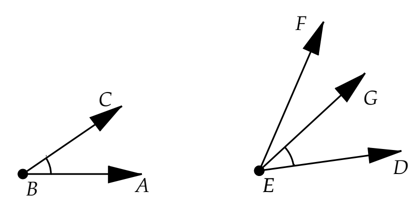



Figure 11.11

Note that this definition does not say anything about angle *measure*. We have not yet defined a way to associate angles with numbers.

We will classify types of angles based on their relationship to right angles.

**Definition 11.18.** *An angle that is less than a right angle is called an* acute *angle. An angle greater than a right angle is called* obtuse.

**Theorem 11.35.** *If an angle is acute then its supplementary angle is obtuse, and vice-versa.*

Proof: Suppose angle $\angle ABC$ is acute and let $\angle ABD$ be a right angle, with $\overrightarrow{BC}$ between $\overrightarrow{BA}$ and $\overrightarrow{BD}$. Let $E$ be a point on the opposite ray to $\overrightarrow{BA}$. We have by Theorem 11.26 that $\overrightarrow{BD}$ is between $\overrightarrow{BC}$ and $\overrightarrow{BE}$. Thus, since $\angle CBD$ is a right angle, we have that the supplementary angle $\angle CBE$ is greater than a right angle and is therefore obtuse. The other half of the proof is similar. □

**Exercise 11.5.1.** *Prove Theorem 11.27.*

**Exercise 11.5.2.** *Prove Theorem 11.29. Hint: Use the fact that there are two supplementary angles to a given vertical angle.*

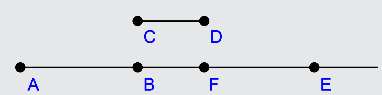
**Exercise 11.5.3.** *Suppose* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{CD} \cong \overline{C'D'}$. *Show that* $\overline{AB} < \overline{CD}$ *if and only if* $\overline{A'B'} < \overline{C'D'}$.

**Exercise 11.5.4.** *Prove Theorem 11.31.*

**Exercise 11.5.5.** *Let* $\Delta ABC$ *be a triangle with all three angles congruent. Show that the triangle must be equilateral.*

Recall that Axiom III-3 is often called the "Segment Addition" axiom. It would be nice to have an actual definition of what it means to add two segments. Here is one attempt.

**Definition 11.19.** *Let* $\overline{AB}$ *and* $\overline{CD}$ *be two segments. Let* $E$ *be a point on* $\overleftrightarrow{AB}$ *with* $A*B*E$ *(axiom II-2). Let* $F$ *be the unique point (given by axiom III-1) on ray* $\overrightarrow{BE}$ *such that* $\overline{CD} \cong \overline{BF}$.




*We then say that* $\overline{AF}$ *is the* sum *of* $\overline{AB}$ *and* $\overline{CD}$ *and we write this as* $\overline{AF} = \overline{AB} + \overline{CD}$.

This is not a perfect definition, for it cannot show that segment addition is commutative. To fix this, one has to define addition in terms of *equivalence classes of segments*. This will be carried out in section **??**. With the above definition, we can still prove a few interesting results.

**Exercise 11.5.6.** *Suppose* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{CD} \cong \overline{C'D'}$. *Show that* $\overline{AB} + \overline{CD} \cong \overline{A'B'} + \overline{C'D'}$.

**Exercise 11.5.7.** *Write down a reasonable definition for the* difference *of two segments* $\overline{AB} - \overline{CD}$, *assuming that* $\overline{AB} > \overline{CD}$.

**Exercise 11.5.8.** *Suppose* $\overline{AB} \cong \overline{CD} + \overline{EF}$. *Show that* $\overline{AB} > \overline{CD}$.

## 11.6 PROJECT 20 - ANGLE ORDER

In this project we will explore properties of the ordering of angles. In this project our exploratory "canvas" will be the canvas of our minds. Feel free to draw diagrams for each new idea, but be careful to make your arguments based *solely* on the theorems and axioms of preceding sections.

Review the theorems that we used to define segment order in the last section. The properties of segment order depended greatly on the "addition" and "subtraction" axioms and theorems for segments. To explore angle ordering we will need the following addition and subtraction theorems for angles.

**Theorem 11.36.** *(Angle Addition) Given* $\overrightarrow{BG}$ *between* $\overrightarrow{BA}$ *and* $\overrightarrow{BC}$, $\overrightarrow{EH}$ *between* $\overrightarrow{ED}$ *and* $\overrightarrow{EF}$, $\angle CBG \cong \angle FEH$, *and* $\angle GBA \cong \angle HED$, *then* $\angle ABC \cong \angle DEF$.

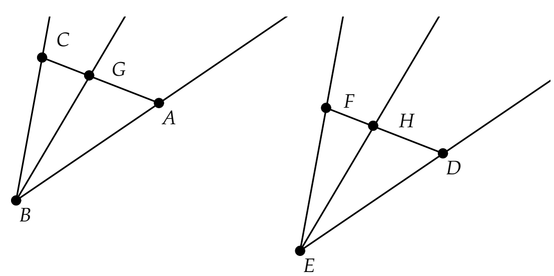



Figure 11.12

Proof: By the Crossbar Theorem we may assume that $G$ is on $\overline{AC}$. (Figure 11.12) Using axiom III-1 we may also assume that $D$, $F$, and $H$ are points chosen on rays $\overrightarrow{ED}$, $\overrightarrow{EF}$, and $\overrightarrow{EH}$ so that $\overline{AB} \cong \overline{ED}$, $\overline{CB} \cong \overline{FE}$, and $\overline{GB} \cong \overline{HE}$.

Using the congruent angles given in the theorem, and the preceding segment congruences, we have by SAS that $\Delta ABG \cong \Delta DEH$ and $\Delta GBC \cong \Delta HEF$. Thus, $\angle DHE \cong \angle AGB$ and $\angle FHE \cong \angle CGB$.

Now, $\angle AGB$ is supplementary to $\angle CGB$, and we know by Theorem 11.28 that the supplement to $\angle DHE$ must be congruent to $\angle CGB$. We already have that $\angle FHE \cong \angle CGB$. By axiom III-4 there is a unique angle on the same side of $\overrightarrow{EH}$ which is congruent to $\angle CGB$. Thus, the supplement to $\angle DHE$ must be $\angle FHE$ and $D$,$H$, and $F$ are collinear.

Since $\overrightarrow{EH}$ is between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, then by Theorem 11.18 we have that $D * H * F$. By the addition axiom (III-3) we have that $\overline{AC} \cong \overline{DF}$. We already know that the angles at $C$ and $F$ are congruent and that sides $\overline{BC}$ and $\overline{EF}$ are congruent. Thus, $\Delta ABC \cong \Delta DEF$ by SAS, and $\angle ABC \cong \angle DEF$. □

**Theorem 11.37.** *(Angle Subtraction) Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\angle CBG \cong \angle FEH$, and $\angle ABC \cong \angle DEF$, then $\angle GBA \cong \angle HED$.*

**Exercise 11.6.1.** *Prove this result by filling in the missing pieces (the places marked why?) in the following proof.*

Proof: We can assume that $\overline{BC} \cong \overline{EF}$, $\overline{AB} \cong \overline{DE}$, $\overrightarrow{BG}$ intersects $\overline{AC}$ at $G$, and $\overrightarrow{EH}$ intersects $\overline{DF}$ at $H$. Then $\overline{CA} \cong \overline{FD}$ and $\angle BCG \cong \angle EFH$. (Why?)

It follows that $\overline{CG} \cong \overline{FH}$, $\overline{BG} \cong \overline{EH}$, and $\angle CGB \cong \angle FHE$. (Why?) The supplements to these last two angles, $\angle BGA$ and $\angle EHD$ are congruent by Theorem 11.28.

Then, $\overline{AG} \cong \overline{DH}$ (Why?) and it follows that $\angle GBA \cong \angle HED$ (Why?). □

Let's review the definition of angle order.

**Definition 11.20.** *$\angle ABC < \angle DEF$ if there exists a point $G$ interior to $\angle DEF$ with $\angle ABC \cong \angle DEG$.*

The next theorem states that angle order satisfies the usual properties. For brevity, we will denote an angle such as $\angle BAC$ be $\angle A$, with the points $B$ and $C$ assumed to be points on the sides.

**Theorem 11.38.** *(Angle Order)*

*(i)* *One and only one of the following holds:* $\angle A < \angle D$, *or* $\angle A \cong \angle D$, *or* $\angle A > \angle D$.

*(ii)* *If* $\angle A < \angle D$ *and* $\angle D \cong \angle G$, *then* $\angle A < \angle G$.

*(iii)* *If* $\angle A > \angle D$ *and* $\angle D \cong \angle G$, *then* $\angle A > \angle G$.

*(iv)* *If* $\angle A < \angle D$ *and* $\angle D < \angle G$, *then* $\angle A < \angle G$.

Note the similarity between this theorem and the one for segment ordering. In fact if we look at the addition, subtraction, and other congruence results for segments and compare these with the corresponding results for angles we see that the results are basically identical, except for changing the word “segment” to “angle” and vice-versa.

Since segments and angles are dual notions, it should not be that difficult to prove this theorem.

**Exercise 11.6.2.** *Prove part (ii) of this result by filling in the missing pieces (the places marked why?) in the following proof.*

Proof: (Proof of part (ii). Proofs of other parts similarly use angle–segment duality).

Let $\angle A = \angle BAC$, $\angle D = \angle EDF$, and $\angle G = \angle HGI$. Since $\angle A < \angle D$, then there is a point $J$ interior to $\angle EDF$ such that $\angle BAC \cong \angle EDJ$. We can assume that $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DJ}$. We can also assume that $J$ is on $\overline{EF}$ (Why?).

We have $\overline{BC} \cong \overline{EJ}$ and $\angle ABC \cong \angle DEJ$ (Why?). By the definition of segment order we have $\overline{BC} < \overline{EF}$.

Since $\angle D < \angle G$, then there is a point $K$ interior to $\angle HGI$ such that $\angle EDF \cong \angle HGK$. Then, $\overline{EF} < \overline{HI}$ and $\angle DEF \cong \angle GHL$ (Why?).

So, we have $\overline{BC} < \overline{HI}$ (Why?). Thus, there is a point $L$ with $H{*}L{*}I$ such that $\overline{BC} \cong \overline{HL}$. Then, since $\angle ABC \cong \angle DEJ (= \angle DEF)$ and $\angle DEF \cong \angle GHL$, we get by SAS congruence that $\Delta ABC \cong DeltaGHL$ and so $\angle BAC \cong \angle HGL$. By definition, then $\angle A < \angle G$. □

Let’s look at how angle ordering can be used to *prove* one of Euclid’s axioms.

**Theorem 11.39.** *(Euclid's Fourth Postulate) All right angles are congruent.*

Proof: Let $\angle BAC$ and $\angle FEG$ be two right angles. If $D$ is opposite of $B$ on $\overrightarrow{AB}$ and $H$ is opposite of $F$ on $\overrightarrow{EF}$, then $\angle BAC \cong \angle DAC$ and $\angle FEG \cong \angle HEG$.

Suppose that $\angle BAC$ is not congruent to $\angle FEG$. By angle ordering, one of these angles is less than the other. We may assume that $\angle FEG$ is less than $\angle BAC$. Then, there is a ray $\overrightarrow{AI}$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $\angle BAI \cong \angle FEG$. (Figure 11.13)

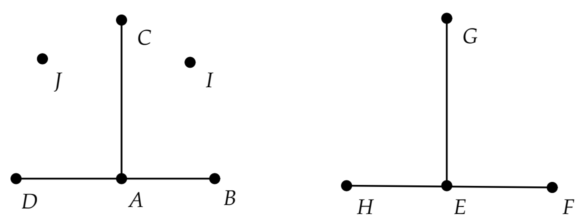



Figure 11.13

Since supplements of congruent angles are congruent, (Theorem 11.28) then $\angle DAI \cong \angle HEG$. By angle transitivity $\angle DAI \cong \angle FEG$.

Since $\angle BAI \cong \angle FEG$ and $\angle FEG < \angle BAC$, then by the previous theorem we have that $\angle BAI < \angle BAC$. Likewise, $\angle BAI < \angle DAC$. Thus, there is a ray $\overrightarrow{AJ}$ between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ such that $\angle BAI \cong \angle DAJ$.

We know by the definition of ordering that $\angle DAC > \angle DAJ$. Now, since $I$ is in the interior of $\angle BAC$ and $B * A * D$ then by part 3 of Theorem 11.20 we know that $C$ will be in the interior of $\angle DAI$. Thus, $\overrightarrow{AC}$ is between $\overrightarrow{AI}$ and $\overrightarrow{AD}$ and $\angle DAC < \angle DAI$.

**Exercise 11.6.3.** *Put all of the preceding results together to create a string of angle inequalities and congruences to show that $\angle DAI > \angle DAI$, and thus reach a contradiction to one of the axioms (which one?)*

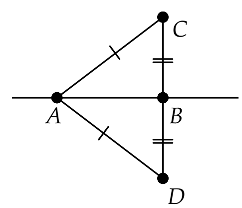
Since we have reached a contradiction, then our original assumption, that $\angle BAC$ is not congruent to $\angle FEG$, must be false and we have proved that $\angle BAC$ is congruent to $\angle FEG$. □

Angle ordering can be used to prove results dealing with the construction of triangles with three given lengths.

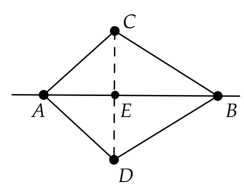
**Theorem 11.40.** *If two points $C$ and $D$ are on opposite sides of a line $\overleftrightarrow{AB}$ and if $\overline{AC} \cong \overline{AD}$ and $\overline{BC} \cong \overline{BD}$ then $\angle ABC \cong \angle ABD$ and $\angle BAC \cong \angle BAD$, and $\Delta ABC \cong \Delta ABD$.*

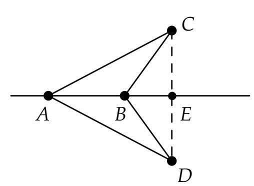
Proof: We may assume that $A \neq B$.

Suppose $B$ is on $\overline{CD}$. Then $\Delta ADC$ is an isosceles triangle. Thus, $\angle ACB \cong \angle ADB$. By SAS we have that $\Delta ABC \cong \Delta ABD$, and thus $\angle ABC \cong \angle ABD$ and $\angle BAC \cong \angle BAD$. The case where $A$ is on $\overline{CD}$ can be handled similarly.




If both $A$ and $B$ are not on $\overline{CD}$ then we have two isosceles triangles $\Delta BDC$ and $\Delta ADC$. Since $C$ and $D$ are on opposite sides of $\overleftrightarrow{AB}$, then $\overline{CD}$ intersects $\overleftrightarrow{AB}$ at some point $E$. We know that $B * A * E$, $A * E * B$, or $A * B * E$. It is clear that $A$ and $B$ are interchangeable in terms of generality of the proof, so we have just two possibilities: $A * E * B$ or $A * B * E$.




**Exercise 11.6.4.** *Use angle addition (and subtraction) to show that $\angle ACB \cong \angle ADB$ in both cases. Then, show that $\Delta ACB \cong \Delta ADB$*

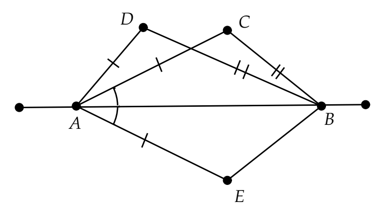
□

The following theorem is essentially Proposition 7 of Book I of *Elements*.

**Theorem 11.41.** *With the same assumptions as the previous theorem, but with $C$ and $D$ on the same side of $\overleftrightarrow{AB}$, then it must be the case that $D = C$. (Informally, this says that there is only one way to construct a triangle with three given side lengths.)*

Proof:

By axiom III-4 there is a unique ray $\overrightarrow{AE}$ on the other side of $\overleftrightarrow{AB}$ from $C$ and $D$ with $\angle CAB \cong \angle BAE$. We can also assume that $\overline{CA} \cong \overline{EA}$. Then, by SAS we have that $\Delta ABC \cong \Delta ABE$.




Then, $\overline{BE} \cong \overline{BC}$ and by the previous theorem (using the transitivity of segment congruence) we have that $\Delta ABD \cong \Delta ABE$ and thus $\Delta ABC \cong \Delta ABD$. But, by axiom III-4 this implies that $D$ must be on $\overrightarrow{AC}$ and also on $\overrightarrow{BC}$. Since these rays already intersect at $C$ then $D = C$. □

We can use the preceding results to prove the following classical triangle congruence, which is also Proposition 8 of Book I of *Elements*.

**Theorem 11.42.** *(SSS) If in two triangles $\Delta ABC$ and $\Delta DEF$ each pair of corresponding sides is congruent then so are the triangles.*

**Exercise 11.6.5.** *Prove SSS. Here is an outline of the proof for you to complete: At point $A$ on $\overline{AB}$, we can construct an angle $\angle BAF'$ with $F'$ on the other side of $\overleftrightarrow{AB}$ from $C$ with $\angle BAF' \cong \angle EDF$ (Why?). We can assume $\overline{AF'} \cong \overline{DF}$ (Why?). Then, $\Delta ABF' \cong \Delta DEF$ (Why?). Also, $\Delta ABF' \cong \Delta ABC$ (Why?).*

### *Project Report*

In this project we have explored the notion of angle order. In your project report provide clear and complete solutions to the exercises. In your conclusion discuss briefly the idea of duality. Explain in your own words what this concept means and why it is such a desirable feature, if present, in an axiomatic system.

## 11.7 CONSTRUCTIONS

The astute reader might be keeping score on how many of Euclid's Propositions have now been put on a solid footing using Hilbert's Incidence, Betweenness, and Congruence axioms. Propositions 2-8, 13, 15, 23, and half of 26 (ASA) have all been verified. Proposition 1 —the construction of equilateral triangles —is still open as it implicitly assumes the intersection of two circles is a continuous process. We will need a new axiom, Dedekind's axiom, to prove this result. Dedekind's axiom will be covered later in this chapter.

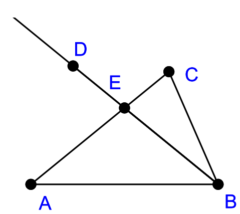
### 11.7.1 Constructions

There are several other construction Propositions in Book I of Euclid's *Elements*. For example, Proposition 9 states that any angle can be bisected. Euclid's proof of this result uses Proposition 1, and thus implicitly assumes the circle intersection property described above. There is another proof of this result that relies only on the existence of isosceles triangles. This approach to Euclid's constructions can be found in Hartshorne's *Geometry: Euclid and Beyond* [11][Chapter 10].

**Theorem 11.43.** *Given a segment $\overline{AB}$ there exists an isosceles triangle with base $\overline{AB}$.*

Proof: By Incidence axiom I-3 we know there is a point $C$ not on $\overline{AB}$. If the angles at $A$ and $B$ are congruent, then the proof is done. So, suppose the angles at $A$ and $B$ are not congruent.

Then, by Theorem 11.38 we know that one of these angles is less than the other. Suppose the angle at $A$ is less than the angle at $B$. Then, there is a ray $\overrightarrow{BD}$ interior to $\angle CBA$ such than $\angle BAD \cong \angle DBA$.



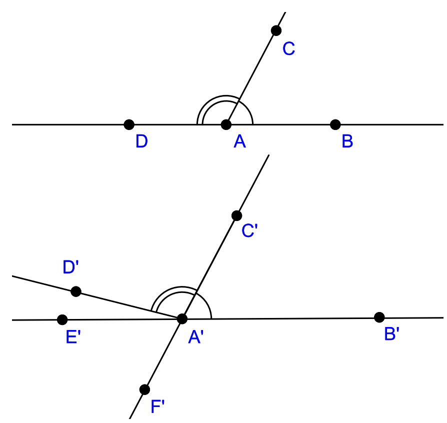

By the Crossbar Theorem (Theorem 11.21), $\overrightarrow{BD}$ must intersect $\overline{AC}$ at a point $E$. Then, the base angles of $\Delta ABE$ are congruent, and so by Theorem 11.31 we have that the triangle is isosceles. We also note for future reference that the point $E$ is *interior* to $\overline{AC}$. $\square$

In order to show the validity of the construction of an angle bisector, we will also need to use the following result about supplementary angles.

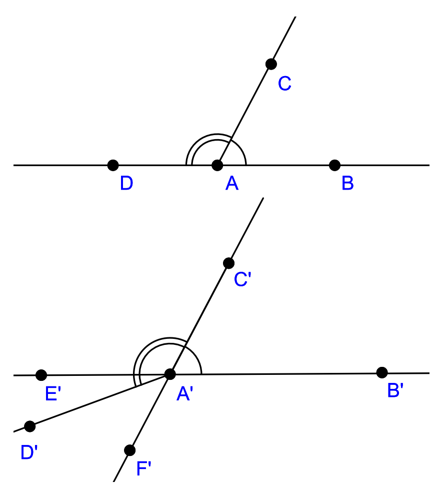
**Theorem 11.44.** *Suppose we have two supplementary angles $\angle BAC$ and $\angle CAD$ on line $\overleftrightarrow{BD}$. Also, suppose that $\angle BAC \cong \angle B'A'C'$ and $\angle CAD \cong \angle C'A'D'$, with $D'$ and $B'$ on opposite sides of $\overleftrightarrow{A'C'}$. Then, angles $\angle B'A'C'$ and $\angle C'A'D'$ are supplementary.*

Proof: We have to show that rays $\overrightarrow{A'D'}$ and $\overrightarrow{A'B'}$ are both on line $\overleftrightarrow{A'B'}$.

Let $E'$ be a point on the ray opposite to $\overrightarrow{A'B'}$. Then, $D'$ and $E'$ are on the same side of $\overleftrightarrow{A'C'}$. Suppose $D'$ is not on $\overrightarrow{A'E'}$. Since $\angle C'A'D'$ is well-defined, then $D'$ cannot be on $\overrightarrow{A'C'}$. By our assumption about $D'$ it cannot be on $\overrightarrow{A'E'}$. If $D'$ is on the same side of $\overleftrightarrow{A'B'}$ as $C'$, then $D'$ is interior to $\angle C'A'E'$ and $\angle C'A'D' < \angle C'A'E'$.




On the other hand, suppose $D'$ is on the opposite side of $\overleftrightarrow{A'B'}$ as $C'$. Let $F'$ be on the opposite ray to $\overrightarrow{A'C'}$. Then $\overrightarrow{A'D'}$ is between $\overrightarrow{A'F'}$ and $\overrightarrow{A'E'}$. By Theorem 11.26 we have that $\overrightarrow{A'E'}$ is between $\overrightarrow{A'D'}$ and $\overrightarrow{A'C'}$. Thus, $\angle C'A'E' < \angle C'A'D'$.



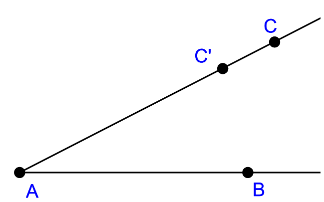

Now, by Theorem 11.28, we have that supplementary angles of congruent angles are congruent. Thus, $\angle C'A'E' \cong \angle CAD$. We are given that $\angle CAD \cong \angle C'A'D'$. By Axiom III-5 we have that $\angle C'A'E' \cong \angle C'A'D'$.

But this contradicts Theorem 11.38, as we cannot have that $\angle C'A'E' \cong \angle C'A'D'$ and either $\angle C'A'E' < \angle C'A'D'$ or $\angle C'A'E' > \angle C'A'D'$. The only conclusion to make is that our assumption about $D'$ is incorrect. That is, we must have that $D'$ is on $\overrightarrow{A'E'}$, and the proof is complete. □ This theorem is essentially a replacement for Euclid's Proposition 14. The statement of this proposition is a bit strange. It specifies when two rays emanating from a common point $P$ must be collinear. Euclid states that they are collinear if another ray from $P$ makes two angles that add to two right angles. The notion of "making two right angles" is not well-defined. In practice, the preceding theorem can serve as a replacement, as it does in the proof of the angle bisector theorem.

**Theorem 11.45.** *(Angle Bisection) Given angle $\angle BAC$ we can find a ray $\overrightarrow{AD}$ between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ such that $\angle BAD \cong \angle DAC$.*

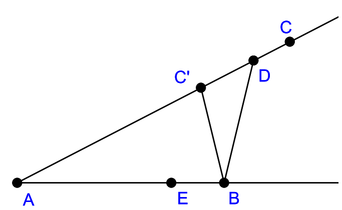
Proof:

By using properties of betweenness and extension of segments, we can assume that $\overline{AC} > \overline{AB}$. On $\overline{AC}$ we can find $C'$ such that $\overline{AB} \cong \overline{AC'}$ ( Axiom III-1).




Following the proof of Theorem 11.43, we can construct an isosceles triangle $\Delta BC'D$ on $\overline{BC'}$, using point $C$. From the construction, we know that $D$ is either on $\overline{CC'}$ or $\overline{BC}$.

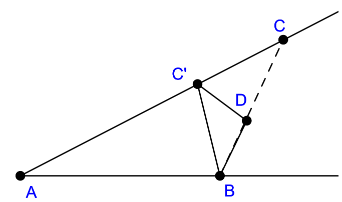
Suppose that $D$ is on $\overline{CC'}$. $\angle BC'D$ is supplementary to $\angle AC'B$. Since $\Delta BC'D$ is isosceles, then by Theorem 11.27 we know that $\angle BC'D \cong \angle DBC'$.




Let $E$ be a point between $A$ and $B$. Then, $A$ and $E$ are on the same side of $\overleftrightarrow{BC'}$. Also, $A$ and $D$ are on opposite sides of $\overleftrightarrow{BC'}$. then, by Theorem 11.12 we have that $D$ and $E$ are on opposite sides of $\overleftrightarrow{BC'}$. Also, since $\Delta ABC'$ is isosceles, then $\angle C'BE \cong \angle AC'B$.

So, we have that $\angle BC'D$ and $\angle AC'B$ are supplementary, $\angle BC'D \cong \angle DBC'$, $\angle AC'B \cong \angle C'BE$, and $D$ and $E$ are on opposite sides of $\overleftrightarrow{BC'}$. By the previous theorem we have that $\angle C'BA$ and $\angle DBC'$ must be supplementary. But, this implies that $D$ is on $\overleftrightarrow{AB}$ which contradicts Incidence axiom I-2.

Now, since $D$ cannot be on $\overline{CC'}$, then it must be on $\overline{BC}$, and so $\overrightarrow{AD}$ is interior to $\angle BAC$. Using SSS congruence on triangles $\Delta DBA$ and $\Delta DC'A$ we have that $\angle BAD \cong \angle DAC'$.




□

Angle bisection is Euclid's Proposition 9. Proposition 10 is the construction of the midpoint of a segment.

**Theorem 11.46.** *Given a segment $\overline{AB}$ there is a point $C$ with $A * C * B$ and $\overline{AC} \cong \overline{CB}$.*

Proof: The proof of this result can be accomplished by using an isosceles triangle construction and is left as an exercise. □

Propositions 11 and 12 of Euclid's *Elements* deal with the construction of perpendiculars.

**Definition 11.21.** *Two lines that intersect are* perpendicular *if one of the angles made at the intersection is a right angle.*

**Theorem 11.47.** *(Proposition 11) Given a line $l$ and a point $P$ on $l$, there exists a line $m$ through $P$ that is perpendicular to $l$.*

Proof: On $l$ one can show that there are points $Q$ and $R$ on opposite sides of $P$ such that $\overline{PQ} \cong \overline{PR}$. It is left as an exercise to prove that this can be done, and then to use an isosceles triangle construction to finish the proof of the result. □

**Theorem 11.48.** *(Proposition 12) Given a line $l$ and a point $P$ not on $l$, there exists a line $m$ through $P$ that is perpendicular to $l$.*

Proof: Let $A$ and $B$ be two distinct points on $l$ (Incidence axiom I-3). Consider $\angle BAP$. By Congruence axiom III-4 we can find a point $Q$ on the other side of $l$ from $P$ such that $\angle BAP \cong \angle QAB$. (Figure11.14)

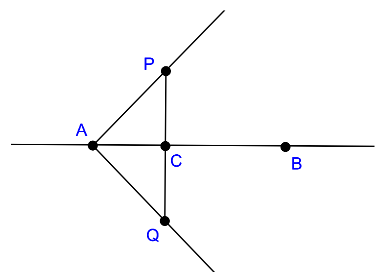



Figure 11.14

We can assume that $\overline{AP} \cong \overline{AQ}$ ( by axiom III-1). Since $P$ and $Q$ are on opposite sides of $l$, then $\overline{PQ}$ intersects $l$ at some point $C$.

If $C \neq A$ then $A$, $C$, and $P$ form a triangle. By SAS congruence, $\Delta ACQ \cong \Delta ACP$ and thus the two angles at $C$ are congruent. Since these two angles are supplementary, the angle at $C$ is a right angle.

If $C = A$, then the angles at $A = C$ are congruent and supplementary. Again, the angle at $C$ will be a right angle. □

**Exercise 11.7.1.** *Prove that segments can be bisected - Theorem 11.46.*

**Exercise 11.7.2.** *Finish the proof of Theorem 11.47.*

**Exercise 11.7.3.** *Prove that a segment $\overline{AB}$ has only one midpoint. [Hint: Suppose it had two. Use betweenness and segment ordering to get a contradiction.]*

**Exercise 11.7.4.** *Prove that an angle has a unique bisector. [Hint: Use the preceding exercise.]*

**Exercise 11.7.5.** *Review the construction of the perpendicular bisector of a segment from Chapter 4. (Section 4.1). Show that this construction is valid and show that there is only one perpendicular bisector of a segment. [Hint: Use uniqueness of midpoints. Then, prove by contradiction using angle ordering.]*

**Exercise 11.7.6.** *Define Line-Circle Continuity as follows: Let $c$ be a circle* *with center $O$ and radius $AB$. If a line $l$ passes through an interior point $P$ of the circle, then it intersects the circle in a point. Assuming this property is true, show that the line intersects the circle at another point. [Hint: There are two cases —the line passes through $O$ or it does not. If it does not, drop a perpendicular from the center of the circle to the line.]*

## 11.8 SEGMENT MEASURE

We have now covered many of the basic results of geometry that are independent of parallel properties. Euclid's Propositions 2-15, 23, and half of 26 (ASA) have now been put on a solid footing using Hilbert's Incidence, Betweenness, and Congruence axioms.

In our development to this point, we have been careful to avoid using one of the most powerful tools of modern geometry - the coordinate based *measure* of segments and angles. The axiomatic development of a geometry robust enough to handle these *analytic* concepts requires the notion of *continuity*. At the most basic level we want to guarantee that lines and segments have no "holes". That is, given a line and a point specified as the origin, we want the ability to reference a point that is 3 units to the right of the origin, or $\pi$ units to the left of the origin.

An axiomatic system for geometry that is based solely on the axioms of incidence, betweenness, and congruence is not sufficient to ensure this idea of continuity. In this section, we will employ a new idea, the notion of *Dedekind cuts*, to ensure the continuous distribution of lengths on lines. The existence of Dedekind cuts will be guaranteed by a new axiom, Dedekind's axiom, from which we can derive all of the necessary continuity principles.

The material in this section is perhaps the most technically challenging of all the topics we will discuss in this chapter. However, this material is perhaps the most intellectually fascinating of the topics we will cover. The question of continuity and the construction of real number measures for segments and angles are among the deepest foundational areas of geometry. We begin our discussion with the statement of Dedekind's axiom.

- **IV-1 (Dedekind's Axiom) If the points on a line $l$ are partitioned into two nonempty subsets $\Sigma_1$ and $\Sigma_2$ (i.e. $l = \Sigma_1 \cup \Sigma_2$) such that no point of $\Sigma_1$ is between two points of $\Sigma_2$ and vice-versa, then there is a unique point $O$ lying on $l$ such that $P_1 * O * P_2$ if and only if one of $P_1$ or $P_2$ is in $\Sigma_1$, the other is in $\Sigma_2$, and $O \neq P_1$ or $P_2$.**

Dedekind's axiom basically says that any splitting of a line into points that are on distinct opposite sides must be accomplished by a unique point $O$ acting as the separator. The pair of subsets described in the axiom is called a *Dedekind cut* of the line.

Dedekind says of this axiom that

> "I think I shall not err in assuming that every one will at once grant the truth of this statement; the majority of my readers will be very much disappointed in learning that by this commonplace remark the secret of continuity is revealed. To this I may say that I am glad if every one finds the above principle so obvious and so in harmony with his own ideas of a line; for I am utterly unable to adduce any proof of its correctness, nor has any one the power. The assumption of this property of the line is nothing less than an axiom by which we attribute to the line its continuity."

Dedekind's axiom certainly does not seem "obvious" at first glance, but upon review it does seem self-evident that two disjoint sets of points on a line that are split into two sides must have a point separating them.

We will use Dedekind's axiom in a variety of ways. Our first application will be in limiting the extremes of "size" of segments. We will need the following notion.

**Definition 11.22.** *We say that segment $\overline{CD}$ is* laid off $n$ times *($n$ a positive integer) on a ray $\overrightarrow{AB}$ if there is a sequence of points $A_0 = A, A_1, A_2, \ldots, A_n$ on $\overrightarrow{AB}$ with $\overline{A_{k-1}A_k} \cong \overline{CD}$ for $k = 1 \ldots n$ and $A * A_k * A_{k+1}$ for $k = 1 \ldots n-1$. We also write $n\overline{CD}$ for laying off $\overline{CD}$ $n$ times.*

Recall that the notation $A * B * C$ is used to designate that point $B$ lies between points $A$ and $C$.

Note that a segment $\overline{CD}$ can always be laid off $n$ times on a ray $\overrightarrow{AB}$. This is a simple consequence of congruence axiom III-1 guaranteeing that we can always continue "copying" $\overline{CD}$ along the ray opposite $\overrightarrow{A_kA_{k-1}}$ at each step of the construction.

The following lemma verifies our intuition as to the ordering of a set of points laid off on a segment.

**Lemma 11.49.** *Let segment $\overline{CD}$ be laid off $n$ times on $\overrightarrow{AB}$. Let $\{A_k\}_{k=0}^n$ be the corresponding sequence of points on $\overrightarrow{AB}$. Then, $A * A_j * A_k$ for all $j = 1 \ldots n-1$, $k = 2 \ldots n$, with $j < k$.*

Proof: Exercise. □

Dedekind's axiom implies Archimedes' axiom which guarantees that no point on a line is infinitely far or infinitely close to a given point.

**Theorem 11.50.** *(Archimedes's axiom) Given $\overline{AB}$ and $\overline{CD}$, there is a positive integer $n$ such that if we lay off $\overline{CD}$ $n$ times on $\overrightarrow{AB}$, starting from $A$, then a point $A_n$ is reached where $A * B * A_n$.*

Proof: Suppose that no such $n$ exists, i.e. for all $n > 1$ the point $A_n$ reached by laying off $\overline{CD}$ $n$ times is not to the "right" of point $B$. Then, for all $n$, the point $A_n \neq B$, for if $A_n = B$ for some $n$, then we could lay off the segment once more and get a point to the right of $B$. Thus, we assume that for all $n$ we have $A * A_n * B$. We will define a Dedekind cut for the line through $A, B$ as follows.

Let $\Sigma_1$ be the set of points $P$ on $\overleftrightarrow{AB}$ such that $A * A_n * P$ for all $n$. Then, $B$ is in $\Sigma_1$ and $\Sigma_1$ is non-empty. Let $\Sigma_2$ be the set of remaining points on $\overleftrightarrow{AB}$. For all $n$, $A_n$ is in $\Sigma_2$ and $\Sigma_2$ is non-empty. Also note that $A$ is in $\Sigma_2$.

We now show that the betweenness condition in Dedekind's axiom is satisfied. Let $Q_1, R_1$ be two points of $\Sigma_1$ and $Q_2, R_2$ be points of $\Sigma_2$. Suppose that $Q_2 * Q_1 * R_2$. Since $Q_2$ and $R_2$ are not in $\Sigma_1$, then for some $n_1$, we must have $A * Q_2 * A_{n_1}$, and for some $n_2$, we have $A * R_2 * A_{n_2}$. If $n_1 = n_2$ we can use the fact that $A * A_{n_2} * A_{n_2+1}$, and four-point betweenness, to show that $A * R_2 * A_{n_2+1}$. Thus, we can assume that $n_1 \neq n_2$, and without loss of generality, that $n_2 > n_1$. By the previous lemma we know that $A * A_{n_1} * A_{n_2}$. Since $A * Q_2 * A_{n_1}$, then $A * Q_2 * A_{n_2}$ by 4-point betweenness. Using $A * Q_2 * A_{n_2}$, $A * R_2 * A_{n_2}$, and $A * A_{n_2} * Q_1$ we have by 4-point betweenness that $A * Q_2 * Q_1$ and $A * R_2 * Q_1$. Thus, $Q_2$ and $R_2$ are on the same side of $Q_1$. But, this contradicts the assumption that $Q_2 * Q_1 * R_2$, and so a point of $\Sigma_1$ cannot be between two points of $\Sigma_2$.

Suppose on the other hand that $Q_1 * Q_2 * R_1$. A similar argument to the previous one will show that $Q_1$ and $R_1$ are on the same side of $Q_2$, which is again a contradiction.

Thus, the conditions for Dedekind's Axiom are satisfied and there must be a unique point $O$ with the properties stated in the axiom. If $O = A_n$ for some $n$, then $A * O * A_{n+1}$ would imply by the axiom that $A_{n+1}$ is in $\Sigma_1$, which is impossible. If $O \neq A_n$, but $A * O * A_k$ for some $k$ then $O$ would be between two points of $\Sigma_2$, which would also contradict Dedekind's axiom. Thus, $O$ must be in $\Sigma_1$.

Now, $O$ is on the same side of $A$ as $A_n$ (for any $n$), for if $O$ was on the other side for some $n$, then $O * A * A_n$, which contradicts the fact that $O$ is in $\Sigma_1$. Also, $\overline{AO} > \overline{CD}$, for if $\overline{AO} < \overline{CD}$, then $A * O * A_1$, and $O$ would be in $\Sigma_2$.

Now, we will show that the existence of point $O$ leads to a contradiction. First, there is a point $X$ with $A * X * O$ and $\overline{XO} \cong \overline{CD}$, (Congruence axiom III-1). Also, $X \neq A_n$ for any $n$ since, if it did match one of the $A_n$, then $O = A_{n+1}$, and $O$ would be in $\Sigma_2$, which is a contradiction. For $P$ in $\Sigma_1$, we have $A * O * P$. Since $A * X * O$, then by 4-point betweenness we have $X * O * P$. By Dedekind's axiom $X$ must be in $\Sigma_2$. Thus, there is an $n > 0$ such that $A * X * A_n$. Since $A * A_n * O$ we have by 4-point betweenness that $X * A_n * O$. By the previous lemma $A * A_n * A_{n+2}$. Thus, by 4-point betweenness we have $A * X * A_{n+2}$. Since $A * A_{n+2} * O$ we have again by 4-point betweenness that $X * A_{n+2} * O$. By segment ordering we have $\overline{XO} > \overline{XA_{n+2}}$. Now, since $A * X * A_n$ and $A * A_n * A_{n+2}$ then by 4-point betweenness we have $X * A_n * A_{n+2}$. Thus, $\overline{XA_{n+2}} > \overline{A_nA_{n+2}}$. By transitivity of segment ordering we have $\overline{XO} > \overline{A_nA_{n+2}}$. But, $\overline{A_nA_{n+2}} > \overline{A_nA_{n+1}}$ and $\overline{A_nA_{n+1}} \cong \overline{CD} \cong \overline{XO}$. Thus, $\overline{XO} > \overline{XO}$. Since a segment cannot be larger than itself we have a contradiction. This completes the proof. □

Note that this theorem implies that no point $B$ can be infinitely far from a fixed point $A$, as we can lay off $\overline{CD}$ a finite number of times to exceed $B$. That is, if we consider $\overline{CD}$ as a unit length we can find $n > 0$ such that $n\overline{CD} > \overline{AB}$. Conversely, if we consider $\overline{AB}$ as a unit length we have $\overline{AB} < n\overline{CD}$ (or $\frac{1}{n}\overline{AB} < \overline{CD}$) and thus there is no infinitely small length.

The following development allows us to talk about the "limit point" of a nested sequence of intervals.

**Definition 11.23.** *A sequence of segments $\overline{A_nB_n}$ $(n = 1, 2, 3 \ldots)$ is called a* nested sequence *if for all $m$ and $n$ we have $A_n * A_{n+1} * B_m$ and $A_n * B_{m+1} * B_m$.*

**Theorem 11.51.** *Let $\overline{A_nB_n}$ be a nested sequence. Then $\overline{A_nB_n} \subset \overline{A_mB_m}$ for all $n > m$. Also, $A_m * A_r * B_n$ and $A_n * B_r * B_m$ for any $n, m$ with $r > m$.*

Proof: Let $C$ be an element of $\overline{A_{m+1}B_{m+1}}$. If $C = A_{m+1}$, then $A_m * C * B_m$ by the definition of a nested sequence and $C$ is an interior element of $\overline{A_mB_m}$. If $C = B_{m+1}$, then $A_m * C * B_m$ by the definition and again $C$ is an interior element of $\overline{A_mB_m}$.

If $C \neq A_{m+1}$, we can assume that $A_{m+1} * C * B_{m+1}$. We are given that $A_{m+1} * B_{m+1} * B_m$. By 4-point betweenness, we have $A_{m+1} * C * B_m$, or $B_m * C * A_{m+1}$. Again, it is given that $B_m * A_{m+1} * A_m$. By 4-point betweenness, we have $B_m * C * A_m$, and $C$ is an interior element of $\overline{A_mB_m}$.

So, $\overline{A_{m+1}B_{m+1}} \subset \overline{A_mB_m}$. Likewise, $\overline{A_{m+2}B_{m+2}} \subset \overline{A_{m+1}B_{m+1}}$, and so $\overline{A_{m+2}B_{m+2}} \subset \overline{A_mB_m}$. Similarly, $\overline{A_{m+k}B_{m+k}} \subset \overline{A_mB_m}$. for all $k > 1$. If we let $n = m + k$ the proof of the first part of the theorem is finished.

For the second part of the theorem we note that $A_m * A_{m+1} * B_n$ and $A_{m+1} * A_{m+2} * B_n$ are true by definition of a nested sequence. Reversing these, we get $B_n * A_{m+2} * A_{m+1}$ and $B_n * A_{m+1} * A_m$. By 4-point betweenness, we then have $B_n * A_{m+2} * A_m$. Again, by definition, $B_n * A_{m+3} * A_{m+2}$. Since $B_n * A_{m+2} * A_m$ we get by 4-point betweenness that $B_n * A_{m+3} * A_m$. Continuing in this fashion, we get that $B_n * A_r * A_m$, or $A_m * A_r * B_n$, for $r > m$. A similar argument can be used to show $A_n * B_r * B_m$ for $r > m$. □

**Theorem 11.52.** *(Cantor's Axiom) Suppose that there is an infinite nested sequence of segments $\overline{A_nB_n}$ $(n > 0)$ on a line $l$. Suppose there does not exist a segment which is less than all of the segments $\overline{A_nB_n}$. Then, there exists a unique point $O$ belonging to all the segments $\overline{A_nB_n}$.*

Proof: We define a Dedekind cut for the line $l$ as follows. Let $\Sigma_1$ consist of all of the points $B_n$ along with any other point $X$ with the property that $A_1 * B_n * X$ for some $n$. Intuitively, this cut consists of all of the right endpoints of the segments in the sequence, along with all of the points to the "right" of these endpoints. Let $\Sigma_2$ consist of all remaining points on the line. We claim that $\Sigma_2$ contains all of the points $A_m$. By the definition of a nested sequence we know that $A_m \neq B_n$ for any $m, n$. Suppose $A_1 * B_n * A_m$ for some $n$. Clearly, $m > 1$. But, $A_1 * B_n * A_m$

contradicts the second claim of the previous theorem, which says that $A_m * A_r * B_n$ for any $n, m$ with $r > m$. In particular, the theorem would imply that $A_1 * A_m * B_n$. Thus, $\Sigma_1$ and $\Sigma_2$ are non-empty sets.

To show that the betweenness property of Dedekind's axiom is satisfied, we first point out that $\Sigma_1$ is entirely contained in $\overrightarrow{A_1B_1}$ and thus we need not consider points in $\Sigma_2$ that lie on the ray opposite to $\overrightarrow{A_1B_1}$. This is because the theorems of section 11.2 show that no point in a ray can be between points on the opposite ray, and vice-versa. Thus, it suffices to consider points $W, X, Y, Z$ with $W, X$ in $\Sigma_1$, $Y, Z$ in $\Sigma_2$, and $Y, Z$ on $\overrightarrow{A_1B_1}$.

We first show that $W * Y * X$ is impossible. We know that $Y$ must satisfy $Y * A_1 * B_n$ or $A_1 * Y * B_n$ for all $n$. Since we assumed $Y$ is not on the ray opposite to $\overrightarrow{A_1B_1}$, we must have $A_1 * Y * B_n$ for all $n$. Since $X$ is in $\Sigma_1$ we have $A_1 * B_n * X$ for some $n$. Since $A_1 * Y * B_n$ and $A_1 * B_n * X$, then by 4-point betweenness, we get $A_1 * Y * X$. (Note: We would likewise have $A_1 * Y * W$.) Now, if $W * Y * X$ then, without loss of generality, we can assume that $A_1 * X * W$ (or $W * X * A_1$). Using 4-point betweenness, we get that $Y * X * A_1$, or $A_1 * X * Y$. This contradicts the fact that $A_1 * Y * X$ and thus $Y$ cannot be between $W$ and $X$.

On the other hand suppose that $Y * W * Z$. We can assume that $A_1 * Z * Y$ (or $Y * Z * A_1$). By 4-point betweenness, we get $Y * W * A_1$, or $A_1 * W * Y$, which contradicts the fact that $A_1 * Y * W$.

Thus, we have constructed a Dedekind cut for $l$ and there must be a unique point $O$ separating $\Sigma_1$ and $\Sigma_2$. We need to show that $O$ belongs to all of the segments $\overline{A_nB_n}$. This is clear since if $A_n$ is in $\Sigma_2$ and $B_n$ is in $\Sigma_1$, then by Dedekind's axiom we have $A_n * O * B_n$.

We are basically done now except to show that $O$ is unique. Suppose there is a second point $O'$ belonging to all of the segments $\overline{A_nB_n}$. Then, $\overline{OO'}$ is contained in all $\overline{A_nB_n}$. (This can be proven using a betweenness argument and is left as an exercise). Let $O''$ be the midpoint of $\overline{OO'}$. Then, $\overline{OO''}$ is less than any of the segments $\overline{A_nB_n}$ which contradicts a hypothesis of the theorem. □

We will now develop a way of *measuring* the length of segments. This development we will make extensive use of the "arithmetic" of segments. We explored the notion of "adding" segments in Definition 11.19 and the exercises following that definition. The following definition will be consistent with this earlier definition, but will make the notion of segment addition more precise.

**Definition 11.24.** *Given segments $a = \overline{AA'}$, $b = \overline{BB'}$, and $c = \overline{CC'}$, we say that $c$ is the* sum *of $a$ and $b$, denoted $c = a+b$, if there exists a point $X$ with $C * X * C'$, $\overline{AA'} \cong \overline{CX}$, and $\overline{BB'} \cong \overline{XC'}$. If we refer to $a+b$, then it is implicitly assumed that there exists a segment $c$ such that $c = a+b$.*

By the properties of segment ordering and addition, we have that all possible choices of $c$ are congruent in this definition. Thus, $a+b$ is well-defined up to congruence, and if we say $c = a+b$, then the equality is defined up to congruence. That is, in this arithmetic, $a+b=c$ means $a+b \cong c$.

**Theorem 11.53.** *Given segments $a, b, c, d$ we have*

*(i)* $a+b = b+a$

*(ii)* $(a+b)+c = a+(b+c)$

*(iii)* *if* $a < b$ *then* $a+c < b+c$

*(iv)* *if* $a < b$ *and* $c < d$ *then* $a+c < b+d$

*(v)* *if* $a = b$ *and* $c = d$ *then* $a+c = b+d$

Proof:Part (i) is clear from the definition.

For part (ii) let $a+b = \overline{PQ}$. Then there is a point $R$ with $P*R*Q$ and $\overline{PR} \cong a$ and $\overline{RQ} \cong b$. Let $S$ be a point with $P*Q*S$ and $\overline{QS} \cong c$. Then, $(a+b)+c$ is congruent to $\overline{PS}$ by definition. It can be shown that $a+(b+c)$ is also congruent to $\overline{PS}$ (exercise). For part (iii) let $a = \overline{AA'}$ and $b = \overline{BB'}$. then, if $a < b$, there must be a point $P$ with $B*P*B'$ and $\overline{AA'} \cong \overline{BP}$. Let $d = \overline{PB'}$. Then, $b = a+d$. So, $b+c = a+d+c = a+c+d$. Thus, since $(a+c)$ is less than $(a+c)+d$, by the definition above, we have that $a+c < b+c$.

For part (iv) we have from part (iii) that $a+c < b+c$. Also, $c+b < d+b$, or equivalently, $b+c < b+d$. Thus, by segment ordering we have $a+c < b+d$.

Part (v) is a simple consequence of the congruence properties of segments. □

**Definition 11.25.** *The* product *of a positive integer $n$ with a segment $a$ is defined as follows: For $n = 1$, let $1a = a$. For $n > 1$ define $na = (n-1)a + a$. This defines the product inductively.*

We will be particularly interested in the arithmetic of *dyadic segments*. These are defined in terms of *dyadic numbers*, numbers of the form $\frac{m}{2^n}$, with $m, n$ integers and $m > 1$, $n \geq 0$.

**Definition 11.26.** *Given a segment $a = \overline{AB}$, construct a sequence of segments $\{s_n\}$ for $n > 0$ as follows: For $n = 0$ let $s_0 = \overline{AB}$. For $n = 1$ let $M_1$ be the midpoint of $\overline{AB}$ and $s_1 = \overline{AM_1}$. For $n = 2$ let $M_2$ be the midpoint of $\overline{AM_1}$ and $s_2 = \overline{AM_2}$. Continue this pattern by letting $M_n$ be the midpoint of $\overline{AM_{n-1}}$ and $s_n = \overline{AM_n}$. Note that $2^n s_n = a$. Define $\frac{1}{2^n}a$ to be the segment $s_n$. Define $\frac{m}{2^n}a = m\frac{1}{2^n}a$, for $m$ a positive integer. Then, $\frac{m}{2^n}a$ will be called a* dyadic segment.

Note that it follows immediately from the definition that $\frac{1}{2^k}(\frac{1}{2^l}a) = \frac{1}{2^{l+k}}a$. Also, $2^k\frac{1}{2^{l+k}}a = 2^k\frac{1}{2^k}(\frac{1}{2^l}a) = \frac{1}{2^l}a$. The following result provides an arithmetic of dyadic segments.

**Theorem 11.54.** *Let $w$ and $v$ be dyadic numbers and $a$ and $b$ segments. Then,*

*(i)* $wa = wb$ *iff* $a = b$.

*(ii)* $w(a + b) = wa + wb$

*(iii)* $(w + v)a = wa + va$

*(iv)* *if* $a < b$ *then* $wa < wb$

*(v)* *if* $w < v$ *then* $wa < va$

*(vi)* *if* $wa < wb$ *then* $a < b$

*(vii)* *if* $wa < va$ *then* $w < v$

Proof: If $w$ and $v$ are positive integers, then the theorem is simply a corollary of Theorem 11.53.

Since $\frac{m}{2^n}a = m\frac{1}{2^n}a$, all we have to show is that the theorem holds for $w = \frac{1}{2^k}$ and $v = \frac{1}{2^l}$.

For part (i) of the theorem, note that if $\frac{1}{2^k}a = \frac{1}{2^k}b$, then $\frac{1}{2^k}a + \frac{1}{2^k}a = \frac{1}{2^k}b + \frac{1}{2^k}b$, by the previous theorem, and so $2\frac{1}{2^k}a = 2\frac{1}{2^k}b$. Continuing to add successively, we get that $2^k\frac{1}{2^k}a = 2^k\frac{1}{2^k}b$. But, $2^k\frac{1}{2^k}a = a$, and likewise for $b$, so $a = b$.

On the other hand, if $a = b$, then by the definition of dyadic segments, we have $wa = wb$.

For part (ii) note that $2^k(w(a+b)) = 2^k\frac{1}{2^k}(a+b) = (a+b)$. Also, $2^k(wa+wb) = (wa+wb)+(wa+wb)+\cdots(wa+wb)$ ($2^k$ times). So,

$$\begin{aligned}
2^k(wa+wb) &= (wa+wa+\cdots+wa)+(wb+wb+\cdots+wb)\\
&= (2^k wa)+(2^k wb)\\
&= a+b
\end{aligned}$$

Thus, $2^k(w(a+b)) = 2^k(wa+wb)$ and by part (i) we have $w(a+b) = wa+wb$.

Part (iii) is left as an exercise.

For part (iv) if $a < b$, then either $wa < wb$, or $wa = wb$, or $wa > wb$. By part (i) $wa = wb$ is impossible. Suppose $wa > wb$. Then, by the previous theorem, we have $wa + wa > wb + wa$ and $wa + wb > wb + wb$. By segment ordering we get $2wa > 2wb$. Likewise, $mwa > mwb$ for $m > 0$. In particular, $2^k wa > 2^k wb$. But, $2^k wa = a$ and $2^k wb = b$. Thus, $a > b$, which is impossible. So, $wa < wb$.

Part (v) is left as an exercise.

Parts (vi) and (vii) follow immediately from (iv) and (v) and the definition of dyadic segments. □

We are now in a position to define segment measure. The measure of a segment will be a *function* that attaches to the segment a real number which will serve as the segment's length. This will allow us to develop our usual notions of analytic geometry.

The development of segment measure necessarily requires an understanding of the axiomatic construction of the real numbers. This construction starts with the natural (or counting) numbers, which are built using the Peano Axioms. A description of the Peano Axioms is given prior to exercise 1.4.9. The natural numbers are then extended to the set of all integers, and the integers are extended to the rationals by looking at equivalence classes of pairs of integers. The task of completing this number system to include all reals can be done in several ways, including Cauchy sequences and the use of Dedekind cuts.

A *Dedekind cut* is a subdivision of the set of all rational numbers into two nonempty, disjoint subsets, say $L$ and $U$, such that no element of $L$ is between two elements of $U$ and vice-versa. We can think of $L$ as a lower interval of the cut and $U$ as an upper interval.

For example, we could let $L$ be the set of rationals whose square is less than 2, and $U$ be the remaining set of rationals. Then, this cut would essentially represent what we consider to be $\sqrt{2}$. In this axiomatic system, the real numbers would consist of all possible Dedekind cuts. With this basic definition, one then has to prove that addition, multiplication, etc, of Dedekind cuts works as we would expect them to. For a complete review of the axiomatic basis for the reals, one can consult any solid real analysis text. For example, the text by Landau [16] does a very thorough job.

It is clear, then, that this axiomatic development of the reals exactly mirrors our use of Dedekind's Axiom in the development to this point of segment arithmetic. It is not surprising that we can use this parallel structure to define segment measure.

**Theorem 11.55.** *Given a segment $u$, which we will call a* unit segment*, there is a unique way of assigning a positive real number, called the* length *and denoted by $\mu(a)$, to any segment $a$, such that for all segments $a$ and $b$ we have*

*(i)* $\mu(a) > 0$ *for all* $a$.

*(ii)* $a \cong b$ *iff* $\mu(a) = \mu(b)$.

*(iii)* $a < b$ *iff* $\mu(a) < \mu(b)$.

*(iv)* $\mu(a + b) = \mu(a) + \mu(b)$.

*(v)* $\mu(u) = 1$.

Proof: Let $\mathcal{D}$ denote the set of all dyadic numbers and let $a$ be any segment. We will split $\mathcal{D}$ into two sets $\mathcal{D}_1^a$ and $\mathcal{D}_2^a$ as follows:

$$w \in \mathcal{D}_1^a \text{ if } wu < a$$
$$w \in \mathcal{D}_2^a \text{ if } wu \geq a$$

These two sets are clearly disjoint, and by the Archimedean axiom, the two sets must be non-empty. Also, no dyadic number in one of the sets can be between two elements of the other. For suppose that $w_2$ of $\mathcal{D}_2^a$ is between $w_1$ and $v_1$ of $\mathcal{D}_1^a$. Then, we can assume $w_1 < w_2 < v_1$. The previous theorem implies that $w_1u < w_2u < v_1u$, but this violates the definition of $\mathcal{D}_1^a$ and $\mathcal{D}_2^a$. Likewise, no element of $\mathcal{D}_1^a$ can be between two elements of $\mathcal{D}_2^a$.

This division of the set of dyadic numbers is thus a Dedekind cut of the set of dyadic numbers (and thus the positive rationals) and must represent a positive real number. Call this number $\mu_a$. Define the length of $a$, denoted $\mu(a)$, to be this number, i.e. $\mu(a) = \mu_a$.

We now have a well-defined function from segments $a$ to the positive real numbers. If $a = u$ then it is clear that the dyadic number 1 will separate $\mathcal{D}_1^u$ and $\mathcal{D}_2^u$. Thus, part (v) of the theorem is proven.

To prove the rest of the theorem it will be helpful to start with part (iv). The number $\mu(a+b)$ is the Dedekind cut corresponding to a segment congruent to $a+b$. That is, we have sets $\mathcal{D}_1^{a+b}$ and $\mathcal{D}_2^{a+b}$ defining the Dedekind cut as described above. We have to show that the number $\mu_a + \mu_b$ satisfies the same conditions for $\mathcal{D}_1^{a+b}$ and $\mathcal{D}_2^{a+b}$ as $\mu(a+b)$ does. That is, we have to show that $\mu_a + \mu_b$ separates the two sets.

By the definition of a Dedekind cut, we know that, for any dyadic numbers $w_1$ and $w_2$, if we have $w_1u < a+b \leq w_2u$, then $w_1 \in \mathcal{D}_1^{a+b}$ and $w_2 \in \mathcal{D}_2^{a+b}$. By Theorem 11.54, $w_1u < a+b \leq w_2u$ is equivalent to $w_1 < \mu(a+b) \leq w_2$. To show that $\mu_a + \mu_b$ separates $\mathcal{D}_1^{a+b}$ and $\mathcal{D}_2^{a+b}$ we must show that if if $w_1 < \mu_a + \mu_b \leq w_2$, then $w_1 \in \mathcal{D}_1^{a+b}$ and $w_2 \in \mathcal{D}_2^{a+b}$. If $w_1 < \mu_a + \mu_b$ then we can find two other dyadic numbers $w_1'$ and $w_1''$ such that $w_1 = w_1' + w_1''$ and $w_1' < \mu_a$, $w_1'' < \mu_b$. Thus, $w_1'$ is in $\mathcal{D}_1^a$ and $w_1''$ is in $\mathcal{D}_1^b$. Equivalently,

$$w_1'u < a \text{ and } w_1''u < b. \tag{11.1}$$

Thus, $w_1u = w_1'u + w_1''u < a + b$, and so $w_1 \in \mathcal{D}_1^{a+b}$. In a similar fashion we can show that if $w_2u \geq a + b$ then $w_2 \in \mathcal{D}_2^{a+b}$. This finishes the proof of part (iv).

To prove part (i) of the Theorem, we note that the Archimedean property states that given $a = \overline{AA'}$ we can find an $n$ such that $na > u$.

Let $p$ be chosen such that $2^p > n$. Now $2^p a > u$, or $\frac{1}{2^p}u < a$. Thus, $\frac{1}{2^p}$ is in $\mathcal{D}_1^a$. Since $\mu(a)$ must separate $\mathcal{D}_1^a$ from $\mathcal{D}_2^a$, we get that $\mu(a) > \frac{1}{2^p} > 0$.

For parts (ii) and (iii) of the theorem, we will prove half of the iff statement first. That is, we will show that $a \cong b$ implies $\mu(a) = \mu(b)$ and $a > b$ implies $\mu(a) < \mu(b)$.

Suppose $a \cong b$. Then, by segment ordering and congruence properties, we have $wu < a$ iff $wu < b$. Thus, the Dedekind cuts for $a$ and $b$ are identical and $\mu(a) = \mu(b)$.

Suppose $a < b$ and let $a = \overline{AA'}$ and $b = \overline{BB'}$. Since $a < b$ there is a point $C$ with $B * C * B'$ and $\overline{AA'} \cong \overline{BC}$. Let $c = \overline{B'C}$. Then $a + c = b$ and by part (iv) we have $\mu(a) + \mu(c) = \mu(b)$. Since $\mu(c) > 0$ we have $\mu(a) < \mu(b)$.

Now, for the other half of statements (ii) and (iii). Suppose $\mu(a) = \mu(b)$. Then either $a \cong b$ or $a < b$ or $a > b$. The last two are impossible, as they imply $\mu(a) < \mu(b)$ or $\mu(a) > \mu(b)$, by the previous section of this proof. Thus, $a \cong b$.

Suppose $\mu(a) < \mu(b)$. Then either $a < b$ or $a \cong b$ or $a > b$. If $a \cong b$ or $a > b$ we again get a contradiction, and thus $a < b$.

Finally, we must show that $\mu$ is unique. Suppose there was another function $\phi$ on segments with the properties of the theorem. Then, $\phi(u) = 1$. Suppose for some segment $a$ that $\phi(a) \neq \mu(a)$. We may assume $\phi(a) < \mu(a)$. There exists a dyadic number $\frac{1}{2^n}$ such that $\phi(a) + \frac{1}{2^n} < \mu(a)$.

Let $b = a + \frac{1}{2^n}u$. Then, $a < b$. Let $c$ be the "difference"of $a$ and $b$, i.e the segment remaining on $b$ that is not congruent to $a$. By the Archimedean axiom we can find $k$ such that $\frac{1}{2^k}u < c$. Also, there is a number $j$ such that $j\frac{1}{2^k}u \leq a$, but $(j+1)\frac{1}{2^k}u \geq a$. Then,

$$a \leq (j+1)\frac{1}{2^k}u = \frac{1}{2^k}u + j\frac{1}{2^k}u < c + a = b$$

Thus, we have found a dyadic number $w = (j+1)\frac{1}{2^k}u$ such that $a \leq wu < b$. Thus, $w = \mu(wu) \geq \mu(a)$. However, we also have

$$w = \phi(wu) < \phi(b) = \phi(a) + \frac{1}{2^n} < \mu(a)$$

which is a contradiction, and $\mu$ must be unique. □

With this theorem we are now at liberty to talk about the "sum" of two segments and to interpret the sum and difference of segments in terms of their numerical lengths. Also, we have that for every positive real number $x$ there is a segment that has length $x$. Finally, we will define the length of $\overline{AB}$ to be zero if $A = B$. It is clear then, that the length of $\overline{AB}$ (denoted by $AB$) is zero if and only if $A = B$.

An important subdivision process that is used extensively in analysis is that of *bisection*. Given a segment $\overline{AB}$, let $A = A_1$ and $B = B_1$. We can find the midpoint $M_1$ of $\overline{AB}$. Choose either $A_2 = M_1$ or $M_1 = B_2$ for the next segment $\overline{A_2B_2}$ in the bisection process. Then, the length of $\overline{A_2B_2}$ is half that of $\overline{A_1B_1}$. Do the bisection process again, yielding segment $\overline{A_3B_3}$ whose length is $\frac{1}{4}$ that of the original segment. It is clear that we can continue this process, generating a sequence of segments $\overline{A_nB_n}$ whose length is $\frac{1}{2^{n-1}}$ times that of the original segment. By the continuity of the reals, this sequence has segments whose lengths approach zero, and thus there must be a point $O$ common to all the sequence terms.

**Exercise 11.8.1.** *Prove Lemma 11.49 by using four-point betweenness.*

**Exercise 11.8.2.** *Let $\overline{AB}$ be a segment and let $C \neq C'$ be two points contained in $\overline{AB}$. Use a betweenness argument to show that the segment $\overline{CC'}$ is contained in $\overline{AB}$.*

**Exercise 11.8.3.** *Let $a$, $b$, and $c$ be three segments. If $a = b + c$, show that $b < a$.*

**Exercise 11.8.4.** *Let $a > b > c$ be three segments. Let $\overline{AA'}$ be a segment congruent to $a$. Show that there are points $B$ and $C$ on $\overline{AA'}$ with $\overline{AB} \cong b$ and $\overline{A'C} \cong c$. Also, show that if $b + c > a$, then $C$ is between $A$ and $B$ and $B$ is between $C$ and $A'$. [Hint: Use Theorem 11.34, Theorem 11.53, and exercise 11.8.3.]*

**Exercise 11.8.5.** *Let $a$, $b$, and $c$ be three segments. Show that $a+(b+c) = (a + b) + c$.*

**Exercise 11.8.6.** *Prove part (iii) of Theorem11.54. [Hint: if $w = \frac{1}{2^k}$ and $v = \frac{1}{2^l}$, find a common denominator for $w + v$ and argue (using the note following the definition of dyadic segments) that one can re-arrange terms to get $(w + v)a = wa + va$.]*

**Exercise 11.8.7.** *Prove part (v) of Theorem11.54. [Hint: If $w = \frac{1}{2^k}$ and $v = \frac{1}{2^j}$, find a common denominator and argue that $w < v$ implies $wa < va$]*

## 11.9 ANGLE MEASURE

To define a measure function for angles, we will need to first establish a Dedekind property for rays.

**Theorem 11.56.** *(Dedekind's Axiom for Rays) Suppose that the rays within $\angle ABC$ are partitioned into two nonempty subsets $\sigma_1$ and $\sigma_2$ such that no ray of $\sigma_1$ is between two rays of $\sigma_2$ and vice-versa. Also suppose that $\sigma_1$ and $\sigma_2$ both contain at least one ray interior to $\angle ABC$. Then there is a unique ray $\overrightarrow{BO}$ interior to $\angle ABC$ such that $\overrightarrow{BO}$ is between two interior rays $\overrightarrow{BP_1}$ and $\overrightarrow{BP_2}$ if and only if one of $\overrightarrow{BP_1}$ or $\overrightarrow{BP_2}$ is in $\sigma_1$, the other is in $\sigma_2$, and $\overrightarrow{BO}$ does not coincide with $\overrightarrow{BP_1}$ or $\overrightarrow{BP_2}$. We will call such a partition a Dedekind cut for $\angle ABC$.*

Proof: Consider segment $\overline{AC}$. We know by earlier work on betweenness that the only rays within $\angle ABC$ are those which intersect $\overline{AC}$, and for each point $D$ on $\overline{AC}$ with $A * D * C$, we know that $\overline{BD}$ is interior to $\angle ABC$. Thus, since $\sigma_1$ and $\sigma_2$ partition the angle, and there is a direct correspondence between rays within the angle and points on $\overline{AC}$, then the intersections of the rays in $\sigma_1$ and $\sigma_2$ will partition the points on $\overline{AC}$.

We define a Dedekind cut on $\overleftrightarrow{AC}$ as follows. First, $\overrightarrow{BC}$ must be in one of $\sigma_1$ or $\sigma_2$. We can assume that it is in $\sigma_2$. Then, $\overrightarrow{BA}$ cannot also be in $\sigma_2$, for we know there is an interior ray in $\sigma_1$ and this ray would then be between two rays of $\sigma_2$, as every interior ray is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. So, $\overrightarrow{BA}$ is in $\sigma_1$. On the line $\overleftrightarrow{AC}$ define $\Sigma_1$ to be the set of intersection points of rays of $\sigma_1$ with $\overline{AC}$ (thus $A$ is in $\Sigma_1$ ) along with the ray opposite to $\overrightarrow{AC}$. Let $\Sigma_2$ be the set of intersection points of rays of $\sigma_2$ with $\overline{AC}$ (thus $C$ is in $\Sigma_2$) along with the ray opposite to $\overrightarrow{CA}$. Then, $\Sigma_1$ and $\Sigma_2$ partition $\overleftrightarrow{AC}$.

Does this cut satisfy the betweenness condition for Dedekind's axiom? Suppose that $X, Y$ are in $\Sigma_1$ and $Z$ is in $\Sigma_2$ with $X * Z * Y$. If $Z$ is not interior to $\overline{AC}$ then it is on the opposite ray to $\overrightarrow{CA}$. But, $X, Y$ are on ray $\overrightarrow{CA}$, and thus it is impossible for $X * Z * Y$. So, $Z$ must be interior to $\overline{AC}$.

If neither $X$ nor $Y$ is interior to $\overline{AC}$, then both would be on the ray opposite $\overrightarrow{AC}$ and $Z$ would also be on this ray, since $X * Z * Y$. This contradicts $Z$ being interior to $\overline{AC}$. Thus, one of $X$ or $Y$ must be interior to $\overline{AC}$. Without loss of generality, we can assume that $X$ is interior to $\overline{AC}$. If $Y$ is not interior to $\overline{AC}$, then $Y$ would be on the ray opposite $\overrightarrow{AC}$. Since $Z$ is on $\overrightarrow{AC}$, we have $Z * A * Y$. Then, using four-point betweenness with $Y * A * Z$ and $Y * Z * X$, we have $A * Z * X$, and thus $Z$ is between two points of $\Sigma_1$, which is impossible. We are forced to conclude that all of $X, Y, Z$ are interior to $\overline{AC}$. But, then $X * Z * Y$ would imply that the rays associated with these points have this betweenness property which is impossible by the hypotheses of the theorem.

An exactly analogous argument rules out the possibility of $X$ in $\Sigma_1$ and $Z, W$ in $\Sigma_2$ with $Z * X * W$.

Thus, there is a unique point $O$ on $\overleftrightarrow{AC}$ with the properties specified by Dedekind's axiom. If $O * A * C$ then $O$ is on the ray opposite $\overrightarrow{AC}$. By Betweenness axiom II-2, there is a point $E$ with $E * O * A$. Then, $O$ would be between two points of $\Sigma_1$ which is impossible. Likewise, it is impossible for $A * C * O$. Thus, the only possibilities left are that $O = A$ or $O = C$ or $A * O * C$.

Suppose that $O = A$. By the hypothesis of the theorem, we know there is a ray, say $\overrightarrow{BP}$, that is interior to $\angle ABC$ and in $\sigma_1$. By the Crossbar theorem, we can assume $P$ is interior to $\overline{AC}$. Then, $P$ is in $\Sigma_1$ and we have $O$ between two points, $A$ and $P$, of $\Sigma_1$ which contradicts Dedekind's axiom. Thus, $O \neq A$. Likewise, we can show that $O \neq C$.

Thus, $A * O * C$, and $\overrightarrow{BO}$ will have the desired properties since the betweenness properties for rays will follow directly from those of points on the segment $\overline{AC}$. □

Note the slight difference in the Dedekind property for angles, as compared to segments. The angle property is valid for a bounded set of angles, while the segment property is defined for a (possibly) unbounded line.

There is an Archimedean property for angles, just as there was for segments. To state this result we need the notion of "laying off" angles on a given ray.

**Definition 11.27.** *We say that angle $\angle DEF$ is* laid off $n$ times *($n$ a positive integer) on a ray $\overrightarrow{BA}$ if there is a sequence of rays $\overrightarrow{BA_0} = \overrightarrow{BA}, \overrightarrow{BA_1}, \overrightarrow{BA_2}, \ldots, \overrightarrow{BA_n}$ with $\angle A_{k-1}BA_k \cong \angle DEF$ for $k = 1 \ldots n$ and ray $\overrightarrow{BA_k}$ between $\overrightarrow{BA}$ and $\overrightarrow{BA_{k+1}}$ for $k = 1 \ldots n-1$. We also write $n\angle DEF$ for laying off $\angle DEF$ $n$ times.*

We will need the following lemma to prove the Archimdean property for angles. This is directly analogous to Lemma 11.49 that was used to prove the Archimedean property for segments. We will use the notation $\overrightarrow{BA} * \overrightarrow{BC} * \overrightarrow{BD}$ to designate that $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BD}$.

**Lemma 11.57.** *Let angle $\angle DEF$ be laid off $n$ times on $\overrightarrow{BA}$. Let $\{\overrightarrow{BA_k}\}_{k=0}^{n}$ be the corresponding sequence of rays. Then, $\overrightarrow{BA} * \overrightarrow{BA_j} * \overrightarrow{BA_k}$ for all $j = 1 \ldots n-1$, $k = 2 \ldots n$, with $j < k$.*

Proof: Exercise. □

**Theorem 11.58.** *(Archimedes Axiom for Angles) Given $\angle ABC$ let $\angle ABM$ be the angle bisector. Then, for any angle $\angle DEF$, there is a positive integer $n$ such that if we lay off $\angle DEF$ $n$ times beginning on $\overrightarrow{BA}$ (yielding $\overrightarrow{BA_n}$) then $\angle ABM < \angle ABA_n$.*

Proof: Suppose that no such $n$ exists, i.e. for all $n > 1$ the angle $\angle ABA_n$ reached by laying off $\angle DEF$ $n$ times is not greater than $\angle ABM$. Then, for all $n$ we have that $\overrightarrow{BA_n} \neq \overrightarrow{BM}$, for if $\overrightarrow{BA_n} = \overrightarrow{BM}$ for some $n$, then we could lay off the angle one more time, yielding an angle that is within $\angle ABC$ and also greater than $\angle ABM$. Thus, we can assume that for all $n$, $\angle ABA_n < \angle ABM$. We will define a Dedekind cut for the angle $\angle ABC$ as follows.

Let $\sigma_1$ be the set of rays $\overrightarrow{BP}$ within $\angle ABC$ such that $\angle ABA_n < \angle ABP$ for all $n$. That is, $\overrightarrow{BA} * \overrightarrow{BA_n} * \overrightarrow{BP}$ for all $n$. Then, $\overrightarrow{BM}$ is in $\Sigma_1$ and $\overrightarrow{BC}$ is also in $\sigma_1$ as $\angle ABM < \angle ABC$. Let $\sigma_2$ be the set of remaining rays within $\angle ABC$. For all $n$, $\overrightarrow{BA_n}$ is in $\sigma_2$ and $\sigma_2$ is non-empty. We note that $\overrightarrow{BA}$ is in $\sigma_2$. Also, both $\sigma_1$ and $\sigma_2$ contain interior points of $\angle ABC$.

Next we show the betweenness condition in Theorem 11.56 is satisfied. Let $\overrightarrow{BQ_1}$, $\overrightarrow{BR_1}$ be rays in $\sigma_1$ and $\overrightarrow{BQ_2}$, $\overrightarrow{BR_2}$ be rays in $\sigma_2$. Suppose that $\overrightarrow{BQ_2} * \overrightarrow{BQ_1} * \overrightarrow{BR_2}$. Since $\overrightarrow{BQ_2}$ and $\overrightarrow{BR_2}$ are not in $\Sigma_1$, then for some $n_1$, we must have $\overrightarrow{BA} * \overrightarrow{BQ_2} * \overrightarrow{BA_{n_1}}$. Also for some $n_2$, we have $\overrightarrow{BA} * \overrightarrow{BR_2} * \overrightarrow{BA_{n_2}}$. If $n_1 = n_2$ we can use the fact that $\overrightarrow{BA} * \overrightarrow{BA_{n_2}} * \overrightarrow{BA_{n_2+1}}$, and four-point betweenness for angles, to show that $\overrightarrow{BA} * \overrightarrow{BR_2} * \overrightarrow{BA_{n_2+1}}$. Thus, we can assume that $n_1 \neq n_2$, and without loss of generality, that $n_2 > n_1$. By the previous lemma we know that $\overrightarrow{BA} * \overrightarrow{BA_{n_1}} * \overrightarrow{BA_{n_2}}$. Since $\overrightarrow{BA} * \overrightarrow{BQ_2} * \overrightarrow{BA_{n_1}}$, then $\overrightarrow{BA} * \overrightarrow{BQ_2} * \overrightarrow{BA_{n_2}}$ by four-point betweenness for angles. Using $\overrightarrow{BA} * \overrightarrow{BQ_2} * \overrightarrow{BA_{n_2}}$, $\overrightarrow{BA} * \overrightarrow{BR_2} * \overrightarrow{BA_{n_2}}$, and $\overrightarrow{BA} * \overrightarrow{BA_{n_2}} * \overrightarrow{BQ_1}$ we have by 4-point betweenness that $\overrightarrow{BA} * \overrightarrow{BQ_2} * \overrightarrow{BQ_1}$ and $\overrightarrow{BA} * \overrightarrow{BR_2} * \overrightarrow{BQ_1}$. Thus, $\overrightarrow{BQ_2}$ and $\overrightarrow{BR_2}$ are both within $\angle BAQ_1$. But, this contradicts the assumption that $\overrightarrow{BQ_2} * \overrightarrow{BQ_1} * \overrightarrow{BR_2}$, and so a ray in $\sigma_1$ cannot be between two rays of $\sigma_2$.

Suppose on the other hand that $\overrightarrow{BQ_1} * \overrightarrow{BQ_2} * \overrightarrow{BR_1}$. A similar argument to the previous one will again yield a contradiction.

Thus, the conditions for Theorem 11.56 are satisfied and there must be a unique ray $\overrightarrow{BO}$ with the properties stated in the theorem. If $\overrightarrow{BO} = \overrightarrow{BA_n}$ for some $n$, then $\overrightarrow{BA} * \overrightarrow{BO} * \overrightarrow{BA_{n+1}}$ would imply by the theorem that $\overrightarrow{BA_{n+1}}$ is in $\sigma_1$, which is impossible. If $\overrightarrow{BO} \neq \overrightarrow{BA_n}$, but $\overrightarrow{BA} * \overrightarrow{BO} * \overrightarrow{BA_k}$ for some $k$ then $\overrightarrow{BO}$ would be between two rays of $\sigma_2$, which would also contradicts Theorem 11.56. Thus, $\overrightarrow{BO}$ must be in $\Sigma_1$.

To finish the proof, we must show that the existence of ray $\overrightarrow{BO}$ leads to a contradiction. This can be done following the last part of the proof of Theorem 11.50 and will be left as an exercise. □

We note that this theorem implies that there is no infinitely small angle or infinitely large angle, just as we had for segments.

We can now construct angle measure. The proof of this result is basically the same as that for segment measure and will be omitted.

**Theorem 11.59.** *Given an angle $\angle UOV$, which we will call a unit angle, there is a unique way of assigning an angle measure, denoted by $\nu(\angle ABC)$, to any angle $\angle ABC$ such that*

*(i)* $\nu(\angle ABC) > 0$ *for all angles $ABC$.*

*(ii)* $\angle ABC \cong \angle DEF$ *iff* $\nu(\angle ABC) = \nu(\angle DEF)$.

*(iii)* *If* $\angle ABC < \angle DEF$ *then* $\nu(\angle ABC) < \nu(\angle DEF)$.

*(iv)* *If $\overrightarrow{BD}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ then* $\nu(\angle ABD) + \nu(\angle CBD) = \nu(\angle ABC)$.

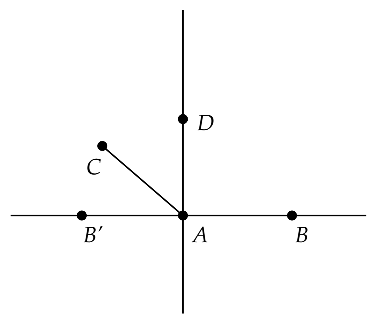
*(v)* $\nu(\angle UOV) = 1$.

Note that it is not important what the unit angle $\angle UOV$ is or what angle measure it is assigned. We could just as well start with a right angle and assign it an angle measure of 90 degrees, or $\frac{\pi}{2}$ or 1 for that matter. Let us assume that the unit angle is a right angle and that it has measure equal to 90 degrees.

**Theorem 11.60.** *The measure of an acute angle is less than* 90*. The measure of any angle is less than* 180*. The measure of an angle plus the measure of its supplement adds to* 180*.*

Proof: The first part of the theorem is clear from the definition of an acute angle. For the second part, the result is clear if the angle is acute or right. Suppose that $\angle BAC$ is obtuse.

Find $B'$ on the opposite ray to $\overrightarrow{AB}$ and $D$ on the same side of the line through $A, B$ such that $\angle BAD$ is a right angle. By the definition of an obtuse angle we have that $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$ and thus the measure of $\angle BAC$ is the sum of the measures of angles $BAD$ and $DAC$.




Now, by Theorem 11.26 we have that $\overrightarrow{AC}$ is between $\overrightarrow{AB'}$ and $\overrightarrow{AD}$. Thus, $\angle DAC$ is acute and the measure of $\angle BAC$ is $90+\beta$, where $\beta < 90$.

For the third part, the result is obvious if the given angle is a right angle. Otherwise, since the supplement of an acute angle is obtuse, and the supplement of an obtuse angle is acute by Theorem 11.35, we can assume that we have an angle $BAC$ that is obtuse and that we can construct the configuration shown in the figure above. Let $\alpha_1$ be the measure of $\angle BAC$, $\alpha_2$ be the measure of $\angle DAC$, and $\alpha_3$ be the measure of $\angle B'AC$. Then, using the same reasoning as in the preceding paragraphs we have that

$$\alpha_1 = 90 + \alpha_2, 90 = \alpha_3 + \alpha_2.$$

Thus, the sum of $\angle BAC$ and its supplement is $\alpha_1 + \alpha_3 = 90 + \alpha_2 + 90 - \alpha_2 = 180$. □ Note that the proceeding result on angles and their supplements is actually Proposition 13 of Book I of *Elements*.

The next result is the converse to the third part of the previous theorem and is also Proposition 14 of Book I of *Elements*.

**Theorem 11.61.** *If the measures of two adjacent angles adds to* 180*, then the angles are supplementary.*

Proof: Let $\angle BAC$ and $\angle CAD$ share side $\overrightarrow{AC}$ and have measure summing to 180. Now, let $\angle CAE$ be the supplement to $\angle BAC$. Since the sum of the measures of $\angle CAE$ and $\angle BAC$ also sum to 180 by the previous theorem, we have that the measure of $\angle CAE$ equals the measure of $\angle CAD$. By the Angle Measure Theorem we have that $\angle CAE \cong \angle CAD$. The result follows by Congruence Axiom III-4. □

At this point in our development of the foundations of geometry, we stop to consider our roster of proven results as compared with Euclid's propositions. We have shown that Euclid's Propositions 2-15, 23, and ASA (half of 26) are on a solid footing using Hilbert's incidence, betweenness, congruence, and Dedekind axioms. We have also shown that segment and angle measure are well defined using these axioms. Hilbert's incidence, betweenness, congruence, and Dedekind axioms form the axiomatic foundation of Euclidean and Hyperbolic geometry, and thus all of our results so far hold in these geometries.

As noted in the section on betweenness (section 11.2), Hilbert's betweenness axioms do not hold in Elliptic geometry. However, in Chapter 14 we will show that the incidence and betweenness axioms can be slightly modified so that the results we have shown so far do hold in Elliptic Geometry.

Also, the results in this section that rely on the betweenness axioms, and the property that a line has infinite extent, will not hold in Elliptic Geometry. For example, Dedekind's axiom does not hold. However, in Elliptic geometry lines are bounded, and thus are equivalent to segments. If we restrict Dedekind's axiom to segments, and recast all of the theorems on segment measure so that they are stated with the assumption that all constructions take place within a segment, then all of the results will still hold. We will cover this in more detail in Chapter 14.

**Exercise 11.9.1.** *Prove Lemma 11.57.*

**Exercise 11.9.2.** *Finish the proof of Theorem 11.58 (Archimedes's Axiom for angles). [Hint: Convert the probelem to one of points on a segment and use the last part of the proof of Theorem 11.50.]*

## 11.10 CONTINUITY

In the preceding sections we have covered many of the basic results that hold in Euclidean, Hyperbolic, and (with some modification) Elliptic geometry. However, we still have not developed a foundation for the very first construction in Euclid's *Elements*, the construction of an equilateral triangle from a given segment. This is Proposition 1 of Book I.

In Euclid's proof of Proposition 1, there is an implicit assumption that a circle with points inside and outside of another circle must intersect that circle somewhere. This basic circle intersection property cannot be proven from the axioms and theorems we have covered so far in this chapter.

To investigate this circle intersection property we first need a definition of circles and their properties.

**Definition 11.28.** *Let $O$ be a point and let $\overline{AB}$ be a segment. The* circle of radius $\overline{AB}$ and center $O$ *is the set of all points $P$ such that $\overline{OP} \cong \overline{AB}$. A point $Q$ is said to be an* interior point *(or said to be* inside *the circle) if $Q = O$ or $\overline{OQ} < \overline{AB}$. If $\overline{OQ} > \overline{AB}$ the point $Q$ is said to be an* exterior *point (or* outside *the circle).*

The basic circle intersection property that we desire can be stated as follows:

- **(Circle-Circle Continuity) Given two circles $c_1$ and $c_2$, with centers $O_1$ and $O_2$, if $c_1$ contains a point inside of $c_2$ and also contains a point outside of $c_2$, then there are exactly two distinct points of $c_1$ that are also on $c_2$. ( We say they *intersect* in two points)**

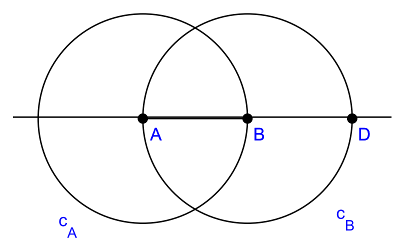
There are several approaches to tackling the issue of circle intersections. One could stipulate that circle-circle intersection is an *axiom*. Alternatively, we could prove the property. Hilbert used Dedekind's axiom to prove the circle-circle intersection property. We will review Hilbert's proof in Chapter 12. This proof is valid in both Euclidean and Hyperbolic geometry. In Chapter 14 we will see that there is also a circle-circle intersection property for Elliptic geometry. Thus, this fundamental continuity property is common to all three of the main geometries we have considered in this text.

We can use the circle-circle intersection property to prove Euclid's Proposition 1.

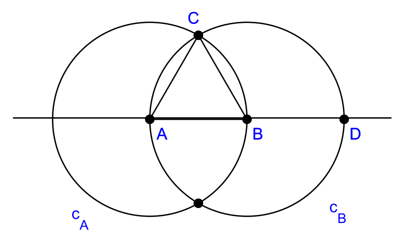
**Theorem 11.62.** *Given $\overline{AB}$ we can construct a triangle $\Delta ABC$ that is an equilateral triangle, i.e. a triangle with all sides congruent.*

Proof: Let $c_A$ be the circle with center $A$ and radius $\overline{AB}$ and let $c_B$ be the circle with center $B$ and radius $\overline{AB}$. Then, $A$ is on $c_B$ and $A$ is inside $c_A$.

On the other side of $B$ from $A$ we can find a point $D$ on $\overrightarrow{AB}$ such that $\overline{BD} \cong \overline{AB}$ (Axiom III-1). Then, $D$ is on $c_B$. Also, $A * B * D$ and $\overline{AD} > \overline{AB}$ . Thus, $D$ is outside $c_A$.




So, $c_B$ has a point $(A)$ inside $c_A$ and a point $(D)$ outside $c_A$. By the Circle-Circle Continuity axiom (IV-1) we have that these two circles intersect in exactly two points. Let $C$ be one of these points.




Since $C$ is on $c_B$ we have that $\overline{BC} \cong \overline{AB}$ and since $C$ is on $c_A$ we have that $\overline{AC} \cong \overline{AB}$. By axiom III-2 we have that $\overline{BC} \cong \overline{AC}$ and $\Delta ABC$ is equilateral. □

## 11.11 TRANSFORMATIONS

A transformation will be some *function* on points. That is, it will be some process whereby points are transformed to other points. This process could be the simple movement of points or could be a more complex alteration of the points. In this section we will investigate transformations that preserve the property of congruence. We will make extensive use of all of the other results covered in this chapter, including segment measure. For sake of exposition and brevity, we will provide proofs without direct reference to Theorem number and verse. Much of the material in this section mirrors that of sections 5.1 and 5.2.

### 11.11.1 Congruence Transformations

**Definition 11.29.** *A function $f$ on the set of all points is called a* congruence transformation *if $f$ has the property that lines are mapped to lines by $f$, and for all points $A$ and $B$ the segment $\overline{AB}$ and the transformed segment $\overline{f(A)f(B)}$ are congruent.*

This simple definition has important implications.

**Theorem 11.63.** *Let $f$ be a congruence transformation. Then*

*(i) $f$ is one-to-one. That is, if $f(A) = f(B)$, then $A = B$.*

*(ii) If $f(A) = A'$ and $f(B) = B'$, then $f$ maps all points between $A$ and $B$ to points between $A'$ and $B'$. That is, $f(\overline{AB}) = \overline{A'B'}$ ($f$ preserves betweenness).*

*(iii) $f$ preserves angles.*

*(iv) $f$ is onto the set of all points. That is, for all points $P'$, there is a point $P$ such that $f(P) = P'$.*

*(v) $f$ preserves parallel lines.*

Proof: (i) Suppose $f(A) = f(B)$. Then, $f(A)f(B) = 0$. Suppose $A \neq B$. Then, $\overline{AB}$ is a segment of non-zero length. By the definition of a congruence transformation, $\overline{AB} \cong \overline{f(A)f(B)}$, and so $\overline{f(A)f(B)}$ would be a segment of non-zero length, which contradicts $f(A)f(B) = 0$. Thus, $A = B$ and $f$ is one-to-one.

(ii) Let $C$ be a point between $A$ and $B$ and let $C' = f(C)$. We need to show that $C'$ is on the line through $A', B'$ and that $C'$ is between $A'$ and $B'$. Since $f$ is one-to-one, $C'$ cannot be $A'$ or $B'$. Now, by segment measure and segment addition, we have that $AB = AC + CB$. Since $f$ is a congruence transformation, we have that

$$A'B' = A'C' + C'B'$$

We know that $C'$ is on the line through $A', B'$, since that is a defining property of $f$. Now either $A'$ is between $B'$ and $C'$, or $B'$ is between $A'$ and $C'$, or $C'$ is between $A'$ and $B'$. In the first case, we would get

$$B'C' = B'A' + A'C'$$

If we subtract this from the equation above, we would get

$$A'B' - B'C' = C'B' - B'A'$$

and

$$2A'B' - 2B'C' = 0$$

So $A'B' = B'C'$, which would contradict the fact that $A'B' < B'C'$ if $A'$ is between $B', C'$. Likewise, we cannot have $B'$ between $A', C'$, and so $C'$ must be between $A', B'$.

(iii) Let $\angle ABC$ be an angle with vertex $B$. Since $f$ preserves segment congruences, by SSS triangle congruence, $\Delta ABC$ and $\Delta f(A)f(B)f(C)$ will be congruent and their angles will be congruent.

(v) By incidence axiom III-3, we know there are at least three distinct points. Since $f$ is one-to-one, there must be two points $A \neq B$ such that $f(A) \neq f(B) \neq P'$. Let $f(A) = A'$ and $f(B) = B'$. There are two cases for $A', B', P'$: either they lie on the same line or not.

If $A', B', P'$ are collinear, then $P'$ is either on the ray $\overrightarrow{A'B'}$ or on the opposite ray. Suppose $P'$ is on $\overrightarrow{A'B'}$. Let $P$ be a point on $\overrightarrow{AB}$ such that $AP = A'P'$. Since $AP = f(A)f(P) = A'f(P)$ and since $P'$ and $f(P)$ are on the same ray $\overrightarrow{A'B'}$, then $P' = f(P)$. If $P'$ is on the opposite ray to $\overrightarrow{A'B'}$, we would get a similar result.

If $A', B', P'$ are not collinear, then consider $\angle P'A'B'$. On either side of the ray through $A, B$, we can find two points $P, Q$ such that $\angle P'A'B' \cong \angle PAB \cong \angle QAB$ (Figure 11.15).

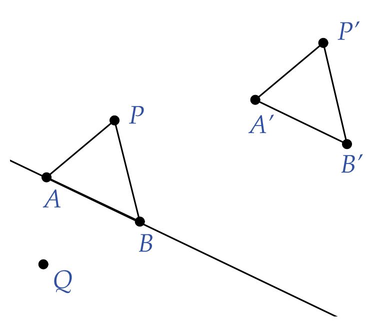



Figure 11.15

We can also choose these points such that $AP = AQ = A'P'$. Since $f$ preserves betweenness (proved in statement (ii)) we know that one of $\overline{A'f(P)}$ or $\overline{A'f(Q)}$ will be on the same side as $\overline{A'P'}$. We can assume that $\overline{A'f(P)}$ is on this same side. Then, since $f$ preserves angles, we have that $\angle P'A'B' \cong \angle f(P)A'B'$ and thus $\overrightarrow{A'P'} \cong \overrightarrow{A'f(P)}$. Since $f$ preserves lengths, we have that $A'P' = AP = f(A)f(P) = A'f(P)$ and thus $P' = f(P)$.

(vi) Let $l, m$ be parallel lines. Suppose that $f(l), f(m)$ were not parallel. Then, for some $P$ on $l$ and $Q$ on $m$, we would have $f(P) = f(Q)$. But, we know that $PQ \neq 0$ as $l$ and $m$ are parallel. Thus, $f(P)f(Q) \neq 0$ and we cannot have $f(P) = f(Q)$. We conclude that $f(l), f(m)$ must be parallel. □

Congruence transformations also have inverses that are congruence transformations.

**Definition 11.30.** *Let $f, g$ be functions on a set $S$. We say that $g$ is the* inverse *of $f$ if $f(g(s)) = s$ and $g(f(s)) = s$ for all $s$ in $S$. That is, the composition of $g$ and $f$ ($f$ and $g$) is the identity function on $S$. We denote the inverse by $f^{-1}$.*

The proof of the inverse property is left as an exercise. It is also the case that the composition of two congruence transformations is again a congruence transformation (exercise).

Congruence transformations can be classified by their fixed points.

**Definition 11.31.** *Let $f$ be a congruence transformation. $P$ is a* fixed point *of $f$ if $f(P) = P$.*

How many fixed points can a congruence transformation have?

**Theorem 11.64.** *If points $A, B$ are fixed by a congruence transformation $f$, then the line through $A, B$ is also fixed by $f$.*

Proof: We know that $f$ will map the line $\overleftrightarrow{AB}$ to the line $\overleftrightarrow{f(A)f(B)}$. Since $A, B$ are fixed points, then $\overleftrightarrow{AB}$ gets mapped back to itself.

Suppose that $P$ is between $A$ and $B$. Then, since $f$ preserves betweenness, we know that $f(P)$ will be between $A$ and $B$. Also

$$AP = f(A)f(P) = Af(P)$$

This implies that $P = f(P)$.

A similar argument can be used in the case where $P$ lies elsewhere on $\overleftrightarrow{AB}$. □

**Definition 11.32.** *The congruence transformation that fixes all points in the plane will be called the* identity *and will be denoted as* id*.*

**Theorem 11.65.** *A congruence transformation $f$ having three non-collinear fixed points must be the identity.*

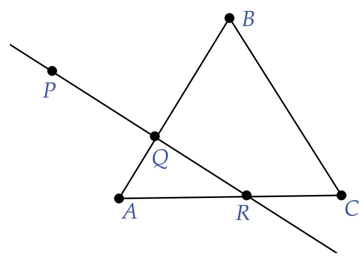



Figure 11.16

Proof: Let $A, B, C$ be the three non-collinear fixed points. From the previous theorem we know that $f$ will fix lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.

Let $P$ be a point not on one of these lines. Let $Q$ be a point between $A, B$ (Figure 11.16). Consider the line through $P, Q$. By Pasch's axiom, this line will intersect one of $\overline{AC}$ or $\overline{BC}$ at some point $R$. By the previous theorem, $f$ fixes the line $\overleftrightarrow{QR}$ and thus fixes $P$. Since $P$ was chosen arbitrarily, then $f$ fixes all points in the plane and is the identity. □

**Corollary 11.66.** *If two congruence transformations $f, g$ agree on any three non-collinear points, then the two functions must agree everywhere, that is, $f = g$.*

The proof of this result is left as an exercise.

It is clear from this theorem that we can classify congruence transformations into three non-trivial (non-identity) types: those with two fixed points, those with one fixed point, and those with no fixed points.

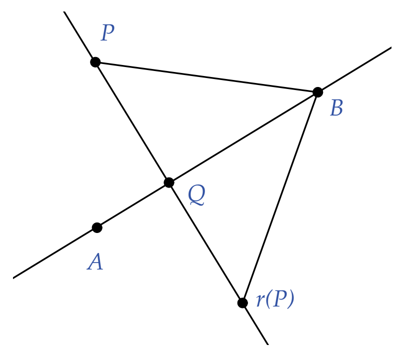
### 11.11.2 Reflections

**Definition 11.33.** *A congruence transformation with two different fixed points, and that is not the identity, is called a* reflection*.*

What can we say about a reflection? By Theorem 11.64 if $A, B$ are the fixed points of a reflection, then the reflection also fixes the line through $A, B$. This line will turn out to be the equivalent of a "mirror" through which the transformation reflects points.

**Theorem 11.67.** *Let $r$ be a reflection fixing $A$ and $B$. If $P$ is not collinear with $A, B$, then the line through $A$ and $B$ will be a perpendicular bisector of the segment connecting $P$ and $r(P)$.*

Proof:




Drop a perpendicular from $P$ to $\overleftrightarrow{AB}$, intersecting at $Q$. At least one of $A$ or $B$ will not be coincident with $Q$; suppose $B$ is not. Consider $\Delta PQB$ and $\Delta r(P)QB$. Since we know that $Q$ and $B$ are fixed points of $r$, then $PQ = r(P)Q$, $BP = Br(P)$, and the two triangles are congruent by SSS.

Since the two congruent angles at $Q$ make up a straight line, $\angle r(P)QB$ will be a right angle and $\overleftrightarrow{AB}$ will be a perpendicular bisector of the segment $\overline{Pr(P)}$. □

We call the line through $A, B$ the *line of reflection* for $r$.

**Theorem 11.68.** *Let $P, P'$ be two points. Then there is a unique reflection taking $P$ to $P'$. The line of reflection will be the perpendicular bisector of $\overline{PP'}$.*

Proof: Let $\overleftrightarrow{AB}$ be the perpendicular bisector of $PP'$ (Figure 11.17). Define a function $r$ on the plane as follows: If a point $C$ is on $\overleftrightarrow{AB}$, let $r(C) = C$. If $C$ is not on this line, drop a perpendicular from $C$ to $\overleftrightarrow{AB}$

intersecting at $Q$, and let $r(C)$ be the unique point on this perpendicular such that $r(C) \neq C$, $Q$ is between $C$ and $r(C)$, and $\overline{r(C)Q} \cong \overline{CQ}$.

Will $r$ be a congruence transformation? We need to show that $r$ maps lines to lines, and that for all $C \neq D$, $r(C)r(D) = CD$. We start with the second condition.

Let $C$ be a point not on $\overleftrightarrow{AB}$ and $D$ a point on the same side of $\overleftrightarrow{AB}$ as $C$. Consider Figure 11.17. By SAS, $\Delta QRD \cong \Delta QR\,r(D)$.

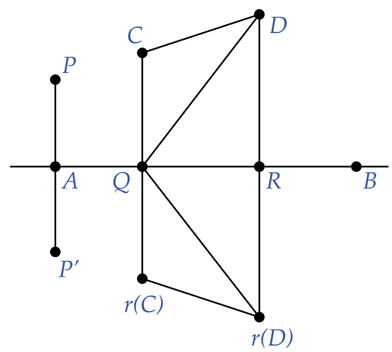



Figure 11.17

Again using SAS congruence, we have $\Delta CQD \cong \Delta r(C)\,Q\,r(D)$. Thus, $CD = r(C)\,r(D)$. Similar arguments using congruent triangles can be used if $D$ is on $\overleftrightarrow{AB}$ or on the other side of $\overleftrightarrow{AB}$ as $C$. (The proof is left as an exercise.)

If $C$ is a point on $\overleftrightarrow{AB}$ and if $D$ is also on $\overleftrightarrow{AB}$, then clearly $CD = r(C)\,r(D)$. If $D$ is not on $\overleftrightarrow{AB}$, then a simple SAS argument will show that $CD = r(C)\,r(D)$.

Now we need to show that $r$ maps lines to lines. It is clear by the definition of $r$ that it maps $\overleftrightarrow{AB}$ to itself. Let $l$ be a line not equal to $\overleftrightarrow{AB}$. Then, $l$ contains a point $C$ not on $\overleftrightarrow{AB}$. If $l = \overleftrightarrow{Cr(C)}$, then, it is clear that all point on $l$ get mapped to the same perpendicular $\overleftrightarrow{Cr(C)}$. There are two remaining cases to consider —either $l$ intersects $\overleftrightarrow{AB}$ (and is not perpendicular to $\overleftrightarrow{AB}$), or $l$ is parallel to $\overleftrightarrow{AB}$.

Suppose that $l$ intersects $\overleftrightarrow{AB}$ at $D$ with $l$ not perpendicular to $\overleftrightarrow{AB}$ (Figure 11.18). Then, $C$, $D$, and $r(C)$ define an angle. Let $E \neq C$ be another point on $\overrightarrow{DC}$.

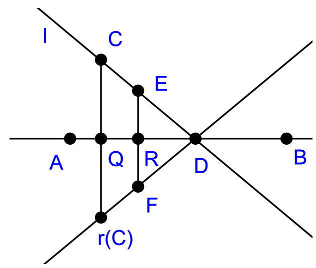



Figure 11.18

On $\overrightarrow{D\,r(C)}$ we can find a point $F$ such that $\overline{DE} \cong \overline{DF}$. By the Crossbar theorem, one of the rays defined by $D$ on $\overleftrightarrow{AB}$ will intersect $\overline{EF}$ at a point $R$. Also, by SAS congruence, we have $\Delta QCD \cong \Delta Q\,r(C)\,D$, where $Q$ is the intersection of the perpendicular from $C$ to $\overleftrightarrow{AB}$. Thus, $\angle QDC \cong \angle QD\,r(C)$. By SAS congruence, we have that $\Delta RDE \cong \Delta RDF$ and $\overline{ER} \cong \overline{FR}$. Thus $\angle FRD \cong \angle ERD$ and $\overleftrightarrow{EF}$ is perpendicular to $\overleftrightarrow{AB}$ at $R$. By the definition of $r$ we conclude that $r(E) = F$. Clearly, the same argument can be used for the ray opposite $\overrightarrow{DC}$. Thus, $r$ maps $l$ to a line through $D$.

The case where $l$ is parallel to $\overleftrightarrow{AB}$ will be left as an exercise.

We have now shown that $r$ is a congruence transformation, but is it unique? Suppose there was another reflection $r'$ taking $P$ to $P'$, where $P$ is not on $\overleftrightarrow{AB}$. By the previous theorem we know that the fixed points of $r'$ are on the perpendicular bisector of $\overline{PP'}$. Since the perpendicular bisector is unique, we have that the fixed points of $r'$ are on $\overleftrightarrow{AB}$. Thus, $r$ and $r'$ have the same values on three non-collinear points $P$, $A$, and $B$ and so $r = r'$. $\square$

We know that a congruence transformation $f$ maps a triangle $\Delta ABC$ to a triangle $\Delta PQR$, with $\Delta ABC \cong \Delta PQR$. We can conversely ask whether, given two congruent triangles, there is a congruence transformation that maps one to the other. We will see that there is such a transformation, given by a sequence of reflections.

Recall that $\Delta ABC \cong \Delta PQR$ if and only if

$$\overline{AB} \cong \overline{PQ}, \overline{AC} \cong \overline{PR}, \overline{BC} \cong \overline{QR}$$

and

$$\angle BAC \cong \angle QPR, \angle CBA \cong \angle RQP, \angle ACB \cong \angle PRQ$$

In other words, there is an *ordering* to the vertex listing for two congruent triangles. It will be important to keep this in mind during the rest of this section.

We will start with an easy case. Clearly, if two triangles are identical, then the identity transformation will map the triangle to itself. The next easiest case is if the triangles share a side.

**Lemma 11.69.** *Let $\Delta ABC \cong \Delta PQR$ with $A = P$ and $B = Q$. Then the triangles are the same, or there is a reflection that takes $\Delta ABC$ to $\Delta PQR$.*

Proof: Suppose that $C$ and $R$ are on the same side of $\overleftrightarrow{AB}$. Then, since there is a unique angle with side $\overline{AB}$ and measure equal to the measure of $\angle BAC$, then $R$ must lie on $\overrightarrow{AC}$. Likewise, $R$ must lie on $\overrightarrow{BC}$. But, the only point common to these two rays is $C$. Thus, $R = C$.

If $C$ and $R$ are on different sides of $\overleftrightarrow{AB}$, then drop a perpendicular from $C$ to $\overleftrightarrow{AB}$, intersecting at $P$. By SAS, $\Delta PAC$ and $\Delta PAR$ are congruent, and thus $\angle APR$ must be a right angle, and $R$ is the reflection of $C$ across $\overleftrightarrow{AB}$. $\square$

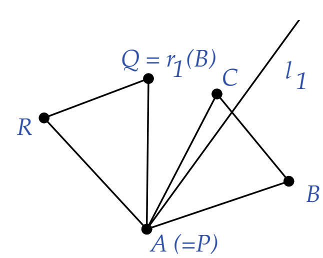
**Lemma 11.70.** *Let $\Delta ABC$ and $\Delta PQR$ be two congruent triangles with $A = P$. Then there is a sequence of at most two reflections that take $\Delta ABC$ to $\Delta PQR$.*

Proof: Clearly, $A \neq Q$. If $B = Q$, then we would be in the case of the previous theorem, and the result follows.

So, we can assume that $A$, $B$, and $Q$ are distinct points. Suppose $Q$, $A$, and $B$ are collinear. Since $Q \neq B$, then $Q$ and $B$ must be on opposite sides of $A$. Let $l_1$ be the perpendicular bisector of $\overline{QB}$. Then, $l_1$ passes through $A$ and reflection across $l_1$ fixes $A$ and maps $B$ to $Q$.

If $Q$, $A$, and $B$ are not collinear, let $l_1$ be the angle bisector of $\angle BAQ$ and $r_1$ the reflection across $l_1$.

Consider $\Delta QAB$. The line $l_1$ will intersect $\overline{QB}$ at some point, say $I$. Then, $\Delta BAI \cong \Delta QAI$ by SAS. This means that $l_1$ is the perpendicular bisector of $\overline{QB}$ and that $Q$ is the reflection of $B$ across $l_1$.



We conclude that if two triangles ($\Delta ABC$ and $\Delta PQR$) share a single point in common ($A = P$), then there is a reflection that maps $\Delta ABC$ to congruent triangle $\Delta AQS$, for some $S$. This triangle is also congruent to $\Delta PQR$, with $P = A$. By the previous theorem, there is at most one reflection that takes $\Delta AQS$ to $\Delta AQR = \Delta PQR$. Thus, there are at most two reflections taking $\Delta ABC$ to $\Delta PQR$. □

We are now ready to consider the general case of two congruent triangles in any configuration.

**Theorem 11.71.** *Let $\Delta ABC$ and $\Delta PQR$ be two congruent triangles. Then there is a sequence of at most three reflections that take $\Delta ABC$ to $\Delta PQR$.*

Proof: Exercise. □

Since the composition of congruence transformations is a congruence transformation, we can re-state this theorem as "Given two congruent triangles, there is a congruence transformation taking one to the other." This theorem has the following amazing corollary:

**Corollary 11.72.** *Every congruence transformation can be written as the product of at most three reflections.*

Proof: Let $f$ be a congruence transformation and consider triangle $\Delta ABC$. We know that $\Delta f(A)\, f(B)\, f(C)$ is a triangle congruent to $\Delta ABC$ and, by the preceding theorem, there is a congruence transformation $g$ composed of at most three reflections taking $\Delta ABC$ to $\Delta f(A)\, f(B)\, f(C)$. Since two congruence transformations that agree on three non-collinear points must agree everywhere, then $f$ must be equal to $g$. □

In this chapter, we have shown that Euclid's Propositions 1-15, 23, and ASA (half of 26) can be put on a solid axiomatic footing using the circle-circle intersection property and Hilbert's incidence, betweenness, congruence, and Dedekind axioms. We have also shown that reflections and their properties are valid within this axiomatic system. The results covered in this chapter form an axiomatic basis for all three of the major geometries we have covered —Euclidean, Hyperbolic, and Elliptic (with some minor adjustments for Elliptic geometry). As such, we will call a geometry based on these axioms a *Universal* geometry.

**Exercise 11.11.1.** *Prove that the composition of two congruence transformations is again a congruence transformation.*

**Exercise 11.11.2.** *Show that any congruence transformation $f$ has an inverse, which we will denote by $f^{-1}$. Also, show that the inverse of a congruence transformation is a congruence transformation.*

**Exercise 11.11.3.** *Prove Corollary 11.66.*

**Exercise 11.11.4.** *Fill in the first open question from the proof of Theorem 11.68 That is, given the definition of the function $r$ from the first part of the proof, let $C$ be a point not on $\overleftrightarrow{AB}$ and let $D$ be a point either on $\overleftrightarrow{AB}$ or on the other side of $\overleftrightarrow{AB}$ as $C$. Show that $r(C)\, r(D) = CD$.*

**Exercise 11.11.5.** *Fill in the first open question from the proof of Theorem 11.68 That is, given the definition of the function $r$ from the first part of the proof, show that $r$ maps the points on a line $l$ parallel to $\overleftrightarrow{AB}$ to points that are all on another line. That is $r$ maps lines parallel to $\overleftrightarrow{AB}$ to other lines.*

**Exercise 11.11.6.** *Prove Theorem 11.71*