CHAPTER 12

Foundations of Neutral Geometry

> The play is independent of the pages on which it is printed, and "pure geometries" are independent of lecture rooms, or of any other detail of the physical world.
>
> – G. H. Hardy in *A Mathematician's Apology* [10] (1877 – 1947)

12.1 TRIANGLES AND PARALLELS

As mentioned at the end of Chapter 11, Hilbert's incidence, betweenness, congruence, and Dedekind axioms form a basis for Euclidean, Hyperbolic, and Elliptic geometry. As such we say these axioms form a basis for *Universal* geometry. As shown in Chapter 11, the theorems of Universal geometry include Euclid's Propositions 1-15, 23, and ASA triangle congruence. Continuing from Proposition 15, there is another set of results that are not universal, but interestingly enough, are still common to two of our geometries —Euclidean and Hyperbolic. Traditionally, the set of results common to these two geometries has been called *Neutral* or *Absolute* geometry.

12.1.1 Exterior Angle Theorem

The first of Euclid's Propositions that does not hold in Universal geometry is the Exterior Angle Theorem (Proposition 16).

Definition 12.1. *Given triangle ΔABC, the angles $\angle CBA$, $\angle BAC$, and $\angle ACB$ are called* interior angles *of the triangle. Their supplementary angles are called* exterior angles*. The two angles of a triangle that are not supplementary to an exterior angle are called* remote interior angles *relative to the exterior angle.*

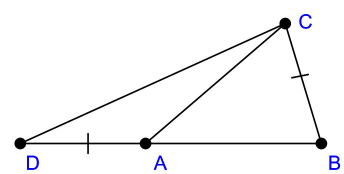
The statement of the Exterior Angle Theorem is as follows:

Theorem 12.1. *(Exterior Angle Theorem) An exterior angle of a triangle is greater than either remote interior angle.*

Let's consider a traditional proof of this theorem:

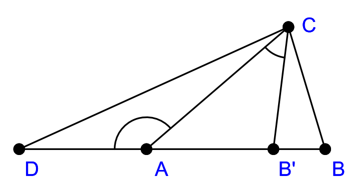
Proof: Let $\angle CAD$ be an exterior angle to $\angle BAC$ in triangle ΔABC. We can assume $\overline{AD} \cong \overline{BC}$. We want to show that $\angle CAD$ is greater than the remote interior angles $\angle ACB$ and $\angle ABC$.

Suppose that $\angle CAD$ was congruent to $\angle ACB$. Triangles ΔCAD and ΔACB have $\overline{AD} \cong \overline{BC}$ and they share side $\overline{AC}$. By SAS these triangles are congruent and $\angle ACD \cong \angle CAB$.



Since $\angle CAD$ and $\angle CAB$ are supplementary, and $\angle CAD \cong \angle ACB$, then the supplementary angle to $\angle ACB$ must be congruent to $\angle CAB$. We are assuming that $\angle CAB \cong \angle ACD$. Thus, by angle transitivity we have that the supplementary angle to $\angle ACB$ is congruent to $\angle ACD$, but since these two angles share a side, then the supplementary angle must be $\angle ACD$. This implies that D lies on $\overleftrightarrow{BC}$, which is impossible. Thus, either $\angle CAD < \angle ACB$ or $\angle CAD > \angle ACB$.

Suppose $\angle CAD < \angle ACB$. If we copy the exterior angle $\angle CAD$ so that one side is on $\overrightarrow{CA}$ and the other is on the same side of $\overrightarrow{CA}$ as B, then the other side will lie in the interior of $\angle ACB$, and thus by the Crossbar Theorem will intersect $\overline{AB}$ at a point B'.



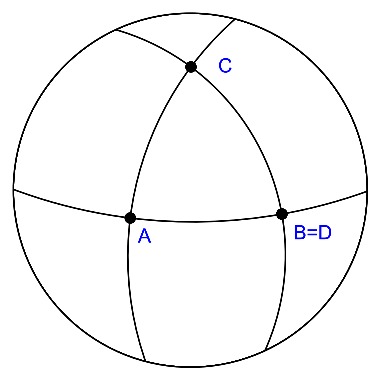
Then, $\angle CAD$ will be an exterior angle of triangle $AB'C$ that is congruent to a remote interior angle, which we just proved is impossible.

Thus, $\angle CAD > \angle ACB$ and a similar argument can be made to show $\angle CAD > \angle ABC$.□

On the face of it, this proof looks solid. It only uses results from earlier sections in this chapter. But, we know this theorem is false in Elliptic geometry, as we showed in chapter 8 that it is possible to construct a triangle with more than one right angle in Elliptic geometry.

So, there must be some subtle assumption that we are making in this proof that is different in Euclidean and Hyperbolic geometry, as compared to Elliptic geometry. The first part of the proof involves a proof by contradiction. We start by assuming that one of the interior angles is congruent to an exterior angle. This leads to the conclusion that point D must be on $\overleftrightarrow{BC}$, which is claimed to be a contradiction. However, this is only a contradiction if we are assuming that lines cannot "double back" on themselves. That is, if lines have infinite extent. However, this is not the case in Elliptic geometry.

Here we have an illustration of a triangle in Elliptic geometry that has three right angles. In chapter 8 we saw that this is possible. In this case, it happens that the point D constructed so that $\overline{AD} \cong \overline{BC}$ will be in exactly the same position as point B, so there is no contradiction!



For Neutral geometry, we will assume the Exterior Angle Theorem holds. Thus, we are implicitly assuming that lines are not bounded, that they are of infinite extent. In Neutral geometry, we then have that Euclid's Propositions 1-15, 23, ASA, and the Exterior Angle Theorem (Proposition 16) all hold. Also, we have all of the results on betweenness, separation, congruence, angle and segment ordering, and numerical measure of segments and angles from Chapter 11. As we have not yet proven the circle-circle intersection property (section 11.10), we will be careful to not use this property (or Euclid's Proposition 1) in this section.

12.1.2 Triangles —Angles and Sides

Let's continue on our review of Euclid's Propositions, assuming that we are working in Neutral geometry. Euclid's version of Proposition 17 states

> In any triangle two angles taken together in any manner are less than two right angles.

We could use angle measure to translate what Euclid means by "less than two right angles" as a comparison to a given angle. However, we will prove the following Theorem, which is equivalent to Euclid's statement.

Theorem 12.2. *(Proposition 17 substitute) In triangle ΔABC, for any pair of angles, say $\angle ABC$ and $\angle BAC$, we have that $\angle ABC$ is less than the supplementary angle to $\angle BAC$.*

The proof of this theorem follows directly from the Exterior Angle Theorem and is left as an exercise.

Propositions 18 and 19 deal with relative comparisons of sides and angles in a triangle.

Theorem 12.3. *(Proposition 18) In a triangle, the larger side is opposite the larger angle.*

Theorem 12.4. *(Proposition 19) In a triangle, the larger angle is opposite the larger side.*

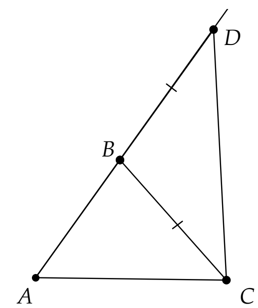
The proofs of these two theorems are left as exercises.

Proposition 20 has traditionally been called the *Triangle Inequality*.

Theorem 12.5. *(Proposition 20) In triangle ΔABC, the sum of two sides of a triangle is always greater than the third side.*

Proof: We have defined segment ordering and addition of segments in section 11.5.2. Without loss of generality, let us assume that the two sides in question are $\overline{AB}$ and $\overline{BC}$. We need to show that the sum of these segments is greater than $\overline{AC}$.

On the ray opposite to $\overrightarrow{BA}$, there is a point D such that $\overline{BD} \cong \overline{BC}$ (Axiom III-1 in section 11.5). Then, B is between A and D and $\overline{AD}$ is the sum of $\overline{AB}$ and $\overline{BD}$. By exercise 11.5.6 we can also say that $\overline{AD}$ is the sum of $\overline{AB}$ and $\overline{BC}$.



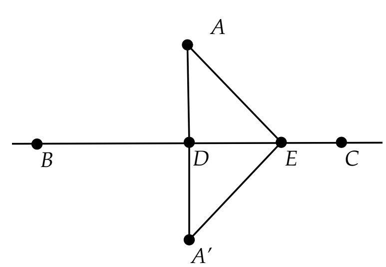
Since B is between A and D, then by Theorem 11.18 and the definition of angle inequality (definition 11.20), we have that $\angle ACD > \angle BCD$. Since ΔCBD is an isosceles triangle we have that $\angle BCD \cong \angle BDC$. Thus, by angle ordering (Theorem 11.38), we have that $\angle ACD > \angle BDC$. By Theorem 12.3 we than have that $\overline{AD} > \overline{AC}$. Since $\overline{AD}$ is the sum of $\overline{AB}$ and $\overline{BC}$, then the sum of $\overline{AB}$ and $\overline{BC}$ is greater than $\overline{AC}$. □

The triangle inequality can be used to prove some very useful results, such as the following:

Theorem 12.6. *Given a line $\overleftrightarrow{BC}$ and a point A not on the line, the perpendicular $\overleftrightarrow{AD}$ from A to a point D on the line has the shortest distance among all segments from A that intersect $\overleftrightarrow{BC}$.*

Proof: Let E be a point not equal to D on $\overleftrightarrow{BC}$.

We can find a point A' on the ray opposite to $\overrightarrow{DA}$ such that $\overline{A'D} \cong \overline{AD}$. Then, triangles ΔADE and $\Delta A'DE$ will be congruent by SAS. Also, $AE + A'E > AA'$ by the Triangle Inequality Theorem. Thus, $2AE > 2AD$ and we are done. □

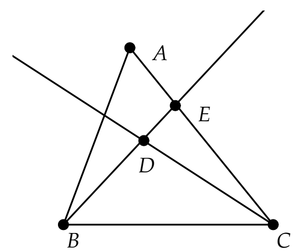


Continuing in our review of Euclid's propositions, we come to Proposition 21. This proposition deals with the relative comparison of side lengths between a triangle and a triangle created in its interior.

Theorem 12.7. *(Proposition 21) Given triangle ΔABC, if we construct triangle ΔDBC with D an interior point of ΔABC, then $BD + CD < BA + CA$. Also, $\angle BDC > \angle BAC$.*

Proof:

Since D is interior to the triangle then $\overrightarrow{BD}$ will intersect side $\overline{AC}$ at some point E with $A * E * C$. In triangle ΔABE we have that $AB + AE > BE$ by the Triangle Inequality theorem.



Then, $AB+AE+EC > BE+EC$. Since $A*E*C$ we have $AE+EC = AC$ and thus $AB + AC > BE + EC$.

Similarly in triangle ΔCDE we can get $CE + ED > CD$. Add BD to both sides and combine to get $CE + BE > CD + BD$.

Then, by segment ordering we have that since $AB + AC > BE + EC$ and $CE + BE > CD + BD$ then $AB + AC > CD + BD$.

The remaining statement about angle inequality is left as an exercise.

□

Euclid's original version of Proposition 22 deals with the construction of a triangle from three given segments. It states that "To construct a triangle from three lengths, it is necessary that when you add any pair of lengths, the sum is greater than the other length."

This is a rather odd statement, as Euclid had already showed in Proposition 20 that if a triangle exists, then the Triangle Inequality holds. Thus, *necessity* had already been proven. In looking at Euclid's proof of Proposition 22, however, we see that he is really proving *sufficiency* of the conclusion. That is, if the sum of any two lengths is greater than the third, than one can construct a triangle with those three lengths. The next theorem will serve as an equivalent statement to Euclid's Proposition 22.

Theorem 12.8. *Given three segments a, b, and c, if the sum of any two of these segments is always greater than the third, then there is a triangle with sides congruent to a, b, and c.*

The proof of this theorem requires the circle-circle intersection property. We will prove this intersection property in the next section, and then we will return to the proof of this proposition. We will be careful not to use this proposition in the remaining proofs of this section.

Continuing in our review of Euclid's axioms, we come to Proposition 23. As mentioned earlier in this section, this is part of Universal geometry, which we have already covered.

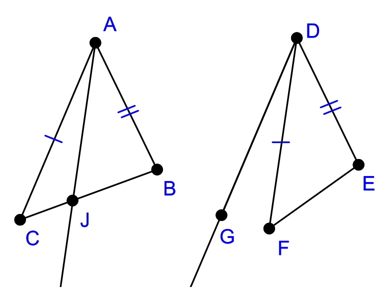
Proposition 24 is another theorem dealing with triangle comparison. It is often called the "Hinge Theorem."

Theorem 12.9. *(Proposition 24) Let ΔABC and ΔDEF be triangles with $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. If $\angle CAB > \angle FDE$, then $\overline{BC} > \overline{EF}$.*

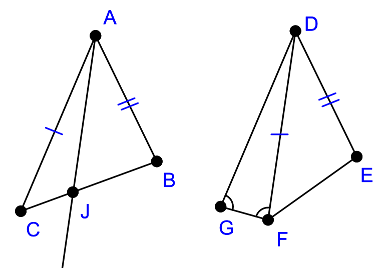
Proof: There are two cases to consider —either $\overline{AC}$ and $\overline{AB}$ are congruent or they are not congruent.

Assume that $\overline{AC}$ and $\overline{AB}$ are not congruent. We can assume that $\overline{AC} > \overline{AB}$.

Since $\angle CAB > \angle FDE$, then we can find a ray $\overrightarrow{AJ}$ interior to $\angle CAB$ such that $\angle JAB \cong \angle FDE$ (definition of angle order). Also, from Congruence Axiom III-4 we can find a ray $\overrightarrow{DG}$ on the other side of $\overrightarrow{DF}$ from $\overrightarrow{DE}$ such that $\angle GDF \cong \angle CAJ$. By addition of angles, $\angle CAB \cong \angle GDE$.



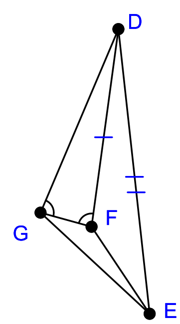
Now, using Congruence Axiom III-1, we can assume that G is the point on $\overrightarrow{DG}$ such that $\overline{DF} \cong \overline{DG}$. Then, ΔDFG is an isosceles triangle and the angles at G and F in ΔDFG are congruent.



We now show that E, F, and G are non-collinear. By Theorem 12.4 we know $\angle BCA < \angle CBA$. Suppose E, F, and G were collinear. Then, by SAS congruence we would have that $\Delta CAB \cong \Delta GDE$, and thus, $\angle BCA \cong \angle EGD$ and $\angle CBA \cong \angle GED$. This implies that $\angle EGD < \angle GED$. But, $\angle EGD \cong \angle GFD$. Thus, we have that the exterior angle $\angle GFD$ would be less than the opposite interior angle $\angle GED$ in ΔDEF which is impossible. We conclude that E, F, and G are non-collinear.

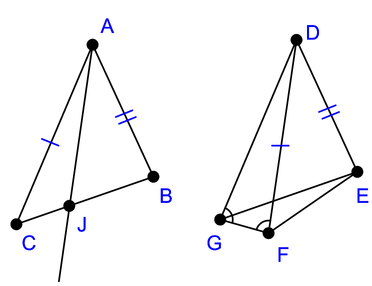
Consider ΔEFG. We now show that $\overrightarrow{GE}$ is interior to $\angle FGD$.

Suppose that $\overrightarrow{GE}$ was exterior to $\angle FGD$. By the Exterior Angle Theorem, $\angle DFG > \angle DEF$. By Theorem 12.4 we know that $\angle DEF > \angle EFD$. Thus, $\angle DFG > \angle EFD$. But, $\angle DFG \cong \angle FGD$. Thus, $\angle FGD > \angle EFD$. Since $\overrightarrow{GE}$ is exterior to $\angle FGD$, then $\angle EGD > \angle FGD$, and so $\angle EGD > \angle EFD$. But this contradicts Theorem 12.7.



We can now assume that $\overrightarrow{GE}$ is interior to $\angle FGD$.

Then $\angle FGE < \angle FGD$. Also, since $\angle DFG \cong \angle FGD$, then $\angle FGE < \angle DFG$. But, $\angle DFG < \angle EFG$. Thus, $\angle FGE < \angle EFG$. By Theorem 12.4 we know that $\overline{EG} > \overline{EF}$. But, by SAS congruence, we know that $\Delta CAB \cong \Delta GDE$ and so $\overline{EG} \cong \overline{BC}$. Thus, $\overline{BC} > \overline{EF}$.



The proof for the case where $\overline{AC} \cong \overline{AB}$ is similar and is left as an exercise. □

Proposition 25 flips the concluding implication from Proposition 24.

Theorem 12.10. *(Proposition 25) Let ΔABC and ΔDEF be triangles with $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. If $\overline{BC} > \overline{EF}$, then $\angle CAB > \angle FDE$.*

Proof: The proof relies on Proposition 24 and is left as an exercise. □

Proposition 26 consists of two triangle congruence results: ASA and AAS. ASA triangle congruence has already been proven when we covered the theorems of Universal geometry. We leave the proof of AAS as an exercise.

12.1.3 Transversals and Parallels

Before we look at Propositions 27 and 28 we need to review some definitions concerning angles and parallels:

Definition 12.2. *A line t is called* transversal *to two other lines l and m if t intersects both lines and the lines are not coincident.*

Let t be transversal to l and m, meeting l at A and m at A' (Figure 12.1). Axiom I-3 guarantees that there is at least one other point on lines l and m and axiom II-2 guarantees that we can choose points B and C on l with $B * A * C$ and B' and C' on m with $B' * A' * C'$. Also, we can assume that B and B' are on the same side of t, and C, C' are also on the same side.

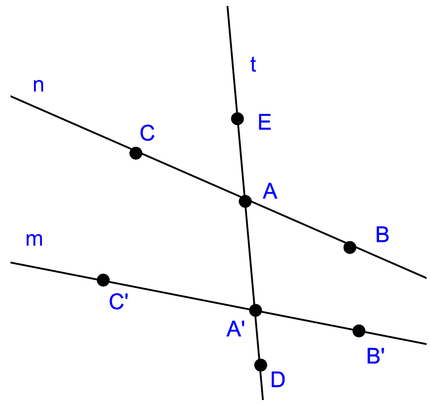


Figure 12.1

Definition 12.3. *Angles* $\angle CAA', \angle C'A'A, \angle BAA'$ *and* $\angle B'A'A$ *are called* interior *angles. (The angles having* AA' *as a side) Also,* $\angle CAA'$ *and* $\angle B'A'A$ *are called* alternate interior angles *as are* $\angle C'A'A$ *and* $\angle BAA'$*. All other angles formed are called* exterior *angles.*

Definition 12.4. *Pairs of angles, one interior and one exterior, on the same side of the transversal, are called* corresponding *angles. For example, in Figure 12.1, angles* $\angle A'AB$ *and* $\angle DA'B$ *are corresponding, as are* $\angle EAC$ *and* $\angle AA'C$*.*

Definition 12.5. *Two lines are* parallel *if they do not intersect.*

We can now consider Euclid's Proposition 27.

Theorem 12.11. *(Alternate Interior Angle Theorem) If alternate interior angles are congruent then the lines are parallel,*

Proof: We are given that $\angle CAA' \cong \angle B'A'A$ as shown in Figure 12.2. Assume that the two lines did meet at some point D. Without loss of generality, we can assume that D is on the same side of t as B and B'.

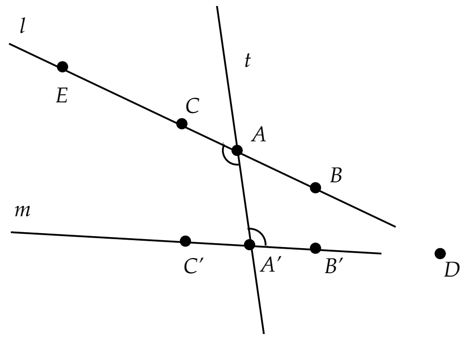


Figure 12.2

There is a point E on $\overrightarrow{AC}$ such that $\overline{AE} \cong \overline{A'D}$. In Neutral geometry we assume that lines cannot double-back on themselves, and so we know that $D \neq E$ and we have two distinct triangles, $\Delta A'AE$ and $\Delta AA'D$. By SAS $\Delta A'AE \cong \Delta AA'D$. Thus, $\angle DAA' \cong \angle EA'A$. Now, $\angle DAA'$ and $\angle EAA'$ are supplements. By Theorem 11.28 we know that since $\angle EAA' \cong \angle DA'A$ then $\angle DAA' \cong \angle C'A'A$. But, the triangle congruence gave $\angle DAA' \cong \angle EA'A$. So, $\angle C'A'A \cong \angle EA'A$. But axiom III-4 on the uniqueness of angles would then imply that E must be on m which is impossible, as then we would have two distinct lines intersecting in more than one point. □

The next two theorems make up Proposition 28.

Theorem 12.12. *If two lines are cut by a transversal so that corresponding angles are congruent then the two lines are parallel.*

Proof: Exercise. □

Theorem 12.13. *If two lines are cut by a transversal so that the interior angles on the same side are supplementary then the two lines are parallel.*

Proof: Exercise. □

These two theorems have the following extremely useful corollaries.

Corollary 12.14. *Two lines that are perpendicular to the same line are parallel.*

Proof: Suppose that lines l and m are both perpendicular to line t. Considering t as a transversal, then the alternate interior angles made are both right angles and hence congruent. Then, the lines are parallel by the Alternate Interior Angle Theorem. □

Note that this implies that the perpendicular dropped from a point to a line must be *unique*, since if there were two lines through the same point that were both perpendicular to the same line, then they would have to be parallel.

Corollary 12.15. *Through a point there is only one line perpendicular to a given line.*

Corollary 12.16. *If l is a line and P is a point not on l, then there is at least one parallel to l through P.*

Proof: By the theorem above there is a unique perpendicular t through P to l. There is also a unique perpendicular m through P to t. Then, t is perpendicular to both l and m and so l and m are parallel. □

Note that this result is Proposition 31. Also note that the parallel through P is not necessarily unique. In fact, Hyperbolic geometry satisfies all of the axioms discussed so far in this chapter, and in this geometry there are multiple parallels through P.

Exercise 12.1.1. *Prove Theorem 12.2. Also, show that this theorem is equivalent to Euclid's version of Proposition 17.*

Exercise 12.1.2. *Prove Theorem 12.4. That is, show that in any triangle, the greater angle lies opposite the greater side. [Hint: Figure 12.3]*

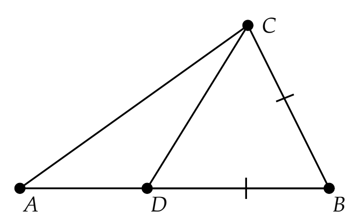


Figure 12.3

Exercise 12.1.3. *Prove Theorem 12.3. That is, show that in any triangle* *the greater side lies opposite the greater angle. [Hint: Assume triangle ABC has $\angle BCA > \angle BAC$. Then, either $\overline{AB} > \overline{BC}$ or $\overline{AB} \cong \overline{BC}$ or $\overline{AB} < \overline{BC}$. Show that two of these are impossible.]*

Exercise 12.1.4. *Prove the AAS triangle congruence result. That is, given triangles ΔABC and ΔDEF if $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$ and $\overline{AC} \cong \overline{DF}$, then $\Delta ABC \cong \Delta DEF$ (Figure 12.4).*

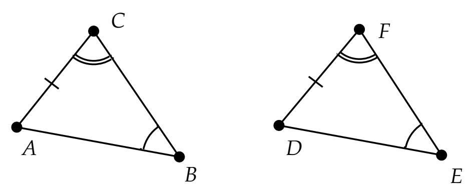


Figure 12.4

[Hint: Suppose that $\overline{BC}$ and $\overline{EF}$ are not congruent. If $\overline{EF} < \overline{BC}$ then there is a point G between B and C with $\overline{GC} \cong \overline{EF}$. Show that this leads to a contradiction of the exterior angle theorem.]

Exercise 12.1.5. *Finish the proof of Theorem 12.7. [Hint: use the Exterior Angle Theorem.]*

Exercise 12.1.6. *Finish the proof of Theorem 12.9. [Hint: There are only two paragraphs in the proof that depend on the two segments $\overline{AC}$ and $\overline{AB}$ being non-congruent. Only a small modification in the argument is neded for each paragraph.]*

Exercise 12.1.7. *Prove Theorem 12.10. [Hint: There are three cases to consider: either $\angle CAB < \angle FDE$, $\angle CAB \cong \angle FDE$, or $\angle CAB > \angle FDE$. Also, use Proposition 24.]*

Exercise 12.1.8. *Prove Theorem 12.12. Hint: Suppose the lines intersect and find a contradiction.*

Exercise 12.1.9. *Prove Theorem 12.13.*

Exercise 12.1.10. *Let ΔABC be a triangle with right angle $\angle ABC$. Call the side opposite the right angle the* hypotenuse *and the other sides* legs *of the triangle. Show that both legs are less than the hypotenuse.*

Exercise 12.1.11. *Let ΔABC be a triangle with right angle $\angle ABC$. On $\overleftrightarrow{AB}$ let D be a point with $D * A * B$. Show that $\overline{BC} < \overline{AC} < \overline{DC}$.*

12.2 CONTINUITY REDUX

In Chapter 11 we discussed (without proof) the Circle-Circle continuity principle:

- **(Circle-Circle Continuity) Given two circles c_1 and c_2, if c_1 contains a point inside of c_2 and also contains a point outside of c_2, then there are exactly two distinct points of c_1 that are also on c_2. (We say they *intersect* in two points)**

There is also a Line-Circle continuity principle:

- **(Line-Circle Continuity) Given a circle c and a line l, if l contains a point inside of c and also contains a point outside of c, then there are exactly two distinct points of c that are also on l.**

In this section we will show that in Neutral geometry, both of these continuity principles can be proven from Dedekind's axiom and the other results we have shown so far. The material in this section is technically dense, much like the material in section 11.8. But, it is worth wading through, as it is crucial to a complete understanding of continuity principles in geometry.

We will start with Line-Circle continuity. We first review some definitions regarding a circle.

Definition 12.6. *The* circle of radius $\overline{AB}$ and center O *is the set of all points X such that $\overline{OX} \cong \overline{AB}$. A point P is said to be an* interior point *(or said to be* inside *the circle) if $P = O$ or $\overline{OP} < \overline{AB}$. If $\overline{OP} > \overline{AB}$ the point P is said to be an* exterior *point (or* outside *the circle).*

Theorem 12.17. *(Line-Circle Continuity) Given a circle c and a line l, if l contains a point inside of c and also contains a point outside of c, then there are exactly two distinct points of c that are also on l.*

Proof: Let O be the center of circle c, let $\overline{AB}$ be the radius, and let P be the point inside c. There are two cases to consider. First, suppose that O is on l. Then, on the two rays on either side of O on l we know by congruence axiom III-1 that we can find unique points I_1 and I_2 with $\overline{OI_1} \cong \overline{AB} \cong \overline{OI_2}$.

Now, suppose that O is not on l. We can drop a perpendicular from O to l which intersects at some point N. Then, N must be an interior point of the circle. This is obvious if $N = P$. If $N \neq P$ then $\angle ONP$ is a right triangle and thus by exercise 12.1.10, we have that $\overline{ON} < \overline{OP}$ and thus N is interior.

We will create a partition for Dedekind's axiom l as follows. Let $\overrightarrow{NT}$ be a ray on l and let $\overrightarrow{NS}$ be the opposite ray (Figure 12.5).

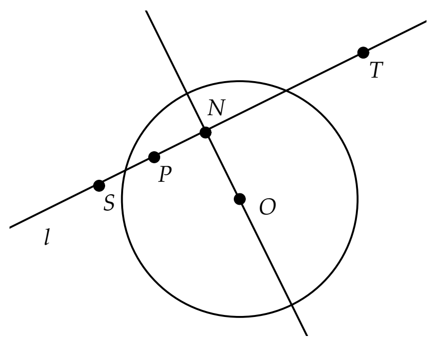


Figure 12.5

Let Σ_1 consist of $\overrightarrow{NT}$ together with all of the points W on l that are inside c (i.e., $\overline{OW} < \overline{AB}$). Let Σ_2 consist of the remaining points of l. Note that Σ_2 consists of points solely on $\overrightarrow{NS}$. Clearly, Σ_1 is not empty. Also, by congruence axiom III-1, we know there is a point R on $\overrightarrow{NS}$ with $\overline{NR} \cong \overline{AB}$. Since ΔONR is a right triangle we have, by exercise 12.1.10, that $\overline{OR} > \overline{NR}$ and R is in Σ_2.

For the betweenness condition of Dedekind's axiom let Q_1, R_1 be two points of Σ_1 and Q_2, R_2 be points of Σ_2. Suppose that $Q_1 * Q_2 * R_1$. There are three cases to consider. First, if Q_1 and R_1 are both on $\overrightarrow{NT}$, then Q_2 cannot be between these two points as it is on the opposite ray. Second, suppose only one of Q_1 and R_1 is on $\overrightarrow{NT}$. We may assume that Q_1 is. Then, $\overline{OR_1} < \overline{AB}$ and R_1, Q_2 are both on the same side of N on l. Now, since $Q_1 * N * Q_2$ and $Q_1 * Q_2 * R_1$ then by four-point betweenness we have $N * Q_2 * R_1$. Since ΔONQ_2 is a right triangle and $N * Q_2 * R_1$ then, by exercise 12.1.11, we have $\overline{OQ_2} < \overline{OR_1}$. But, $\overline{OR_1} < \overline{AB}$, and so $\overline{OQ_2} < \overline{AB}$ which contradicts the fact that Q_2 is in

Σ_2. Lastly, suppose both Q_1 and R_1 are on the ray $\overrightarrow{NS}$ with $\overline{OQ_1} < \overline{AB}$ and $\overline{OR_1} < \overline{AB}$. Then, since $Q_1 * Q_2 * R_1$, we can use exercise 12.1.11 to show that either $\overline{OQ_2} < \overline{OQ_1}$ or $\overline{OQ_2} < \overline{OR_1}$. In either case, $\overline{OQ_2} < \overline{AB}$ which contradicts Q_2 being in Σ_2.

Now, suppose that $Q_2 * Q_1 * R_2$. This is impossible if Q_1 is on $\overrightarrow{NT}$, as Q_2 and R_2 are on the opposite ray. So, $\overline{OQ_1} < \overline{AB}$ and Q_1 is on $\overrightarrow{NS}$. Since $Q_2 * Q_1 * R_2$, then one of Q_2 and R_2 is on the same side of Q_1 as N. We can assume that Q_2 is on the same side of Q_1 as N. Then, either $N * Q_2 * Q_1$ or $N * Q_1 * Q_2$. We cannot have $N * Q_2 * Q_1$, as N is in Σ_1 and we have already shown that a point of Σ_2 cannot be between two points of Σ_1. Thus, we must have $N * Q_1 * Q_2$ (or $Q_2 * Q_1 * N$). From $Q_2 * Q_1 * R_2$, we conclude that N and R_2 are on the same side of Q_1. Thus, either $Q_1 * R_2 * N$ or $Q_1 * N * R_2$. Again, we have shown above that $Q_1 * R_2 * N$ is impossible. Since Q_1 and R_2 are both on $\overrightarrow{NS}$, we cannot have $Q_1 * N * R_2$.

Thus, by Dedekind's axiom there is a unique point X separating Σ_1 and Σ_2. We claim that $\overline{OX} \cong \overline{AB}$, and thus X is an intersection point of l with c.

For suppose that $\overline{OX} < \overline{AB}$. By segment addition and congruence axiom II-1, we can find a point V on the vector opposite to $\overrightarrow{XN}$ with $\overline{XV} + \overline{OX} \cong \overline{AB}$. (see figure 12.6) By the triangle inequality we have $OV < XV + OX = AB$.

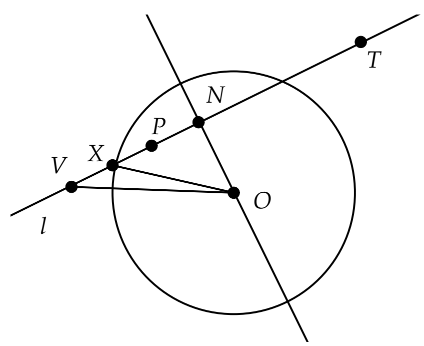


Figure 12.6

Thus, V is in Σ_1. But, this would imply that X is between two points of Σ_1, namely V and N, which is impossible. A similar argument shows that $\overline{OX} > \overline{AB}$ is not possible.

Thus, $\overline{OX} \cong \overline{AB}$. It is left as an exercise to show that X must be on $\overrightarrow{NS}$.

Will this intersection point be unique? Assume that there is another point of intersection X' of ray $\overrightarrow{NS}$ with the circle. Let M be the midpoint of $\overline{XX'}$. Then, triangles ΔOXM and $\Delta OX'M$ are congruent by SSS congruence and thus $\angle OMX \cong \angle OMX'$ and both must be right angles. But, we know by theorem 12.15 and its corollaries that the perpendicular from O to l must be unique, which implies that $N = M$. But, this is impossible since if X and X' are both on the same side of N, then all points on $\overline{XX'}$ are on that side of N. Thus, X is unique.

To finish the proof we need to find an intersection point along the ray $\overrightarrow{NT}$. We leave this as an exercise. □

It is clear from the proof that we only used the point P that was inside the circle in our argument for the first intersection. We thus have the following:

Corollary 12.18. *Given a circle c and a line l, if l contains a point P inside of c, then there are exactly two distinct points of c that are also on l. Also, the intersection points occur on each of the two opposite rays on l defined by P.*

The proof of the second statement of this corollary is left as an exercise. We also have the following result about segments and circles:

Corollary 12.19. *Given a circle c and a segment $\overline{PQ}$, if P is inside c and Q is outside c, then there is a point X interior to $\overline{PQ}$ that is also on c.*

Proof: Exercise. □

Before we prove the circle-circle continuity property, we will take a little side-trip to show that Dedekind's axiom can be extended to arcs of circles.

Definition 12.7. *Let c be a circle with center O. If $A \neq B$ are two points on c then we call the segment $\overline{AB}$ a* chord *of the circle. If a chord passes through the center O we say it is a* diameter *of the circle.*

Definition 12.8. *A chord $\overline{AB}$ of a circle c will divide the points of a circle c into two parts, those points of c on one side of $\overleftrightarrow{AB}$ and those on the other side. Each of these two parts is called an* open arc *of the circle. An open arc determined by a diameter is called a* semi-circle*. If we include the endpoints A and B, we call the arc or semi-circle* closed.

Definition 12.9. *One of the two open arcs defined by a (non-diameter) chord $\overline{AB}$ will be within the angle $\angle AOB$. This will be called a* minor *arc. The arc that is exterior to this angle is called a* major *arc.*

In order to extend Dedekind's axiom to arcs of circles we need to have a notion of betweenness for points on arcs.

Definition 12.10. *Let $\overline{AB}$ be a chord of a circle c with center O and let σ be the minor arc of the chord. Let P, Q, R be points on σ. Then we say that R* is between *P* and *Q if the ray $\overrightarrow{OR}$ is between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. (see figure 12.7)*

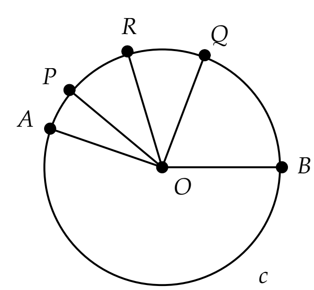


Figure 12.7

Theorem 12.20. *Dedekind's axiom and the Archimedean property can both be extended to minor arcs of circles.*

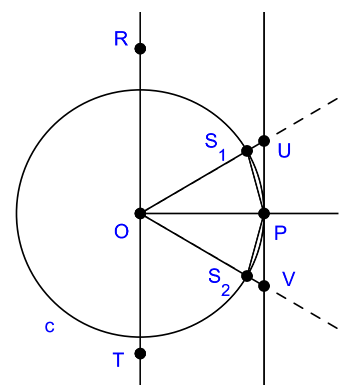
Proof: The definition of betweenness for points on a minor arc is that of betweenness for rays in an angle. There is a direct correspondence between points on a minor arc and rays interior to the angle defined by the chord of the arc. What this correspondence allows us to do is to substitute any statement about points on an arc, including betweenness properties of such points, to statements about rays in an angle and betweenness properties of rays. Since we have already proven extensions of Dedekind's Axiom and the Archimedean property for rays within an angle, then we automatically have these same properties for points on a minor arc. □

We now return to the task of proving circle-circle continuity. We will need the following result for the proof.

Lemma 12.21. *Let P be a point on circle c with center O. Let $\overleftrightarrow{OR}$ be perpendicular to $\overleftrightarrow{OP}$ at O. Then, for any number $s > 0$ there are points S_1 and S_2 on c such that $PS_1 < s$ and $PS_2 < s$. Also, S_1 and S_2 are on opposite sides of $\overleftrightarrow{OP}$ and the same side of $\overleftrightarrow{OR}$.*

Proof: By Theorem 11.55 (Segment measure), there is a segment $\overline{AB}$ having length s. At P construct the perpendicular to $\overleftrightarrow{OP}$.

By congruence axiom III-1, we know there are points U and V on opposite sides of P on this perpendicular such that $\overline{PU} \cong \overline{AB} \cong \overline{PV}$. By congruence axiom III-1, there are points S_1 and S_2 on $\overrightarrow{OU}$ and $\overrightarrow{OV}$, respectively, such that S_1 and S_2 are on c.



Also, $\angle OPU$ and $\angle OPV$ are right angles, so $\angle POU$ and $\angle POV$ are acute (Proposition 17). Then, $\angle POS_1 < \angle POR$. Let $\angle POT$ be the supplementary angle to $\angle POR$. Then, $\angle POS_2 < \angle POT$ (angle order properties with supplementary angles). Thus, S_1 and S_2 are on opposite sides of $\overleftrightarrow{OP}$ and the same side of $\overleftrightarrow{OR}$.

Now, ΔOPS_1 is an isosceles triangle. Thus, $\angle OS_1P$ must be acute (Proposition 17). Then, $\angle PS_1U$ is obtuse, by Theorem 11.35. In triangle ΔPS_1U we than have $\angle PUS_1$ is acute (Proposition 17). Thus, by

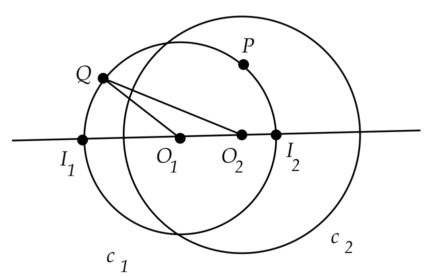
Theorem 12.3, we have that $\overline{PS_1} < \overline{PU}$. Since the length of $\overline{PU}$ is s, we have $PS_1 < s$. A similar argument shows that $PS_2 < s$. □

We will also use the following result to put circle-circle continuity into a "standard" configuration.

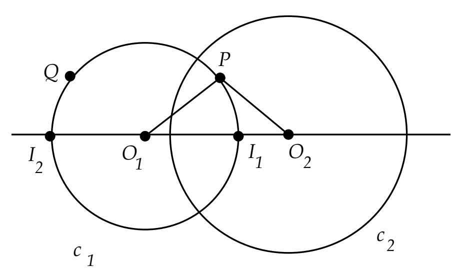
Theorem 12.22. *Let c_1, c_2 be two circles with centers O_1, O_2. Suppose there is a point P on c_1 that is inside c_2 and there is a point Q on c_1 that is outside c_2. Then, the line through the centers O_1, O_2 must meet circle c_1 in two points I_1 and I_2 with one of these points inside c_2 and the other outside c_2.*

Proof: For our analysis, we will use the segment measure properties we covered in Chapter 11. Specifically, we will let r_1 be the measure of the radius for c_1, and r_2 the measure for the radius of c_2. Since O_1 is inside of c_1 then by exercise 12.2.4 we have that $\overleftrightarrow{O_1O_2}$ intersects c_1 in two points I_1 and I_2. There are three cases to consider for point I_1. Either $I_1 * O_1 * O_2$ or $O_1 * I_1 * O_2$ or $O_1 * O_2 * I_1$.

Case (1): $I_1 * O_1 * O_2$. Then, $\overline{I_1O_2} \cong \overline{I_1O_1} + \overline{O_1O_2}$, or $I_1O_2 = O_1O_2 + r_1$. Also, by the triangle inequality for ΔO_1QO_2 we have $O_2Q < O_1O_2 + r_1$. Since Q is outside c_2, we have $O_2Q > r_2$. Then, $r_2 < (O_1O_2 + r_1)\ (= I_1O_2)$. Thus, I_1 is outside c_2.



Case (2): $O_1 * I_1 * O_2$. Then, $O_1O_2 = O_1I_1 + I_1O_2 = r_1 + I_1O_2$. Also, by triangle inequality for ΔO_2PO_1 we have $O_1O_2 < r_1 + O_2P$. Thus, $r_1 + I_1O_2 < r_1 + O_2P$ and so $I_1O_2 < O_2P$. Since $O_2P < r_2$ we have that $I_1O_2 < r_2$ and I_1 must lie inside c_2.



Case (3): $O_1 * O_2 * I_1$. A similar argument to case (2) shows that I_1 must lie inside of c_2 (exercise).

What about I_2? In Case (1), we have $I_1 * O_1 * O_2$. Also, $I_1 * O_1 * I_2$, as I_1 and I_2 must be on opposite sides of O_1 (by the work we did in the proof of Theorem 12.17). Thus, O_2 and I_2 are on the same side of O_1 and we have either $O_1 * I_2 * O_2$ or $O_1 * O_2 * I_2$, In both cases, we can replace I_1 by I_2 in the work above. By doing so, we would either be in Case (2) or (3) for I_2, and so I_2 must be inside c_2.

Now consider Case (2) or (3) for I_1. We have $O_1 * I_1 * O_2$ or $O_1 * O_2 * I_1$. Again we have $I_1 * O_1 * I_2$. By four-point betweenness, in either case we get that $I_2 * O_1 * O_2$. Thus, for I_2, we would be in Case (1) and I_2 must be outside c_2. □

We are now ready to prove circle-circle continuity.

Theorem 12.23. *(Circle-Circle Continuity) Let c_1, c_2 be two circles with centers O_1, O_2. Suppose that c_1 has one point P inside and one point Q outside of c_2. Then, the two circles intersect in two points.*

Proof: By Theorem 12.22 we can assume there is a diameter $\overline{I_1 I_2}$ of c_1 with I_1 outside c_2 and I_2 inside c_2 (Figure 12.8).

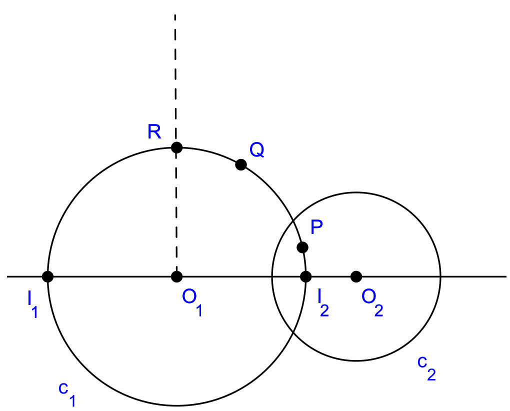


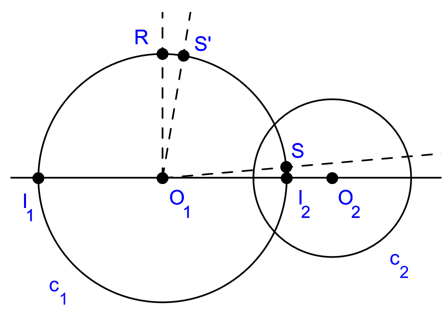
Figure 12.8

By Theorem 11.47, we can construct a perpendicular $\overleftrightarrow{O_1 R}$ to $\overleftrightarrow{I_1 I_2}$ at O_1. Let $\overrightarrow{O_1 R}$ be one of the two rays defined. Then, by congruence axiom III-1 we can assume R is a point such that the measure of $\overline{O_1 R}$ is r_1, where r_1 is the measure of the radius of c_1. That is, R is on c_1. Now, R is either outside c_2, inside c_2, or on c_2 (exercise). If it is on c_2, we have found an intersection point for the two circles. Let this point be M for reference later in this proof. We note that M is on one side of $\overleftrightarrow{I_1I_2}$.

If R is inside c_2, then the points I_1 and R form a pair of inside-outside points to c_2. If R is outside c_2 then the points R and I_2 form a pair of inside-outside points. In either case, we have a well-defined angle, with vertex at O_1, with sides being two rays intersecting c_1 at points inside and outside c_2. Without loss of generality we assume that R is outside c_2 and consider angle $\angle I_2O_1R$.

Since I_2 is inside c_2, then $O_2I_2 < r_2$. Let $s = r_2 - O_2I_2$.

Using Lemma 12.21 for c_1 and the number $\frac{s}{2}$, we have that there is a point S interior to arc $\sigma = arc(I_2R)$ such that $I_2S < \frac{s}{2}$. Likewise, since R is outside c_2 then $O_2R > r_2$. Let $s' = O_2R - r_2$. Then, there is a point S' on this arc such that $RS' < \frac{s'}{2}$.

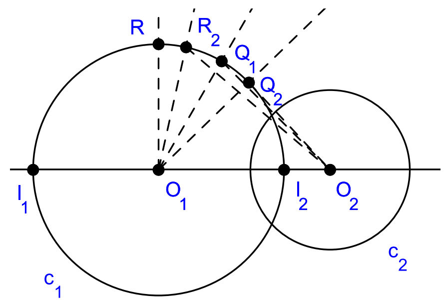


By the triangle inequality, we have that $O_2S < I_2S + O_2I_2$. Since $I_2S < \frac{s}{2} < s < r_2 - O_2I_2$, we have $O_2S < r_2 - O_2I_2 + O_2I_2 = r_2$. Thus, S is inside c_2. Again, by the triangle inequality, we have that $O_2R < RS' + O_2S'$. Since $RS' < \frac{s'}{2} < s'$, then $O_2S' > O_2R - RS' > O_2R - \frac{s'}{2} > O_2R - s' = r_2$. Thus, S' is outside c_2.

Define a Dedekind cut for the rays within $\angle I_2O_1R$ as follows. Let Σ_1 consist of rays with points on σ inside circle c_2. Let Σ_2 consist of rays with points on σ that are on or outside circle c_2. Clearly, these two sets are non-empty. We now show these sets satisfy the betweenness property for Dedekind's axiom for arcs.

Suppose we had a ray $\overrightarrow{O_1Q_1}$ of Σ_1 that was between two rays $\overrightarrow{O_1Q_2}$ and $\overrightarrow{O_1R_2}$ of Σ_2.

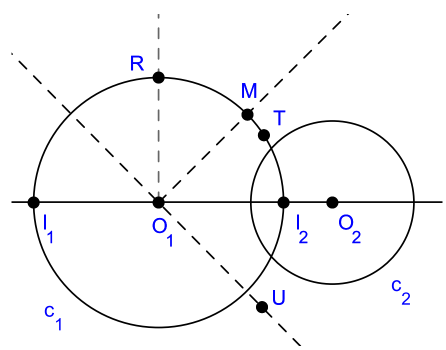
Then, $\angle I_2O_1Q_1$ is greater than $\angle I_2O_1R_2$ (or $\angle I_2O_1Q_2$). In either case, we can show, using Theorem 12.9 that this implies that $\overline{O_2Q_1} > \overline{O_2R_2}$ (or $\overline{O_2Q_1} > \overline{O_2Q_2}$) (exercise). But, this would mean that Q_1 is outside c_2, which contradicts Q_1 being in Σ_1.



Similarly, we cannot have a point of Σ_2 between two points of Σ_1.

Dedekind's property for angles (Theorem 11.56) requires a partition of rays (which we have), a betweenness condition (shown above), and the condition that both subsets of the partition contain at least one interior ray to the angle. These are $\overrightarrow{O_1S}$ and $\overrightarrow{O_1S'}$ defined above. Thus, there is a unique ray $\overrightarrow{O_1M}$ interior to $\angle I_2O_1R$ separating Σ_1 and Σ_2. We need to show that $O_2M = r_2$, where r_2 is the radius of c_2.

Suppose $O_2M > r_2$. Let $t = O_2M - r_2$. Let $\overleftrightarrow{O_1U}$ be perpendicular to $\overleftrightarrow{OM}$ at O. We can choose U so that U is on the other side of $\overleftrightarrow{O_1O_2}$ as M. By Lemma 12.21, there is a point T on c_1 such that T is interior to $\angle UO_1M$ and $MT < \dfrac{t}{2}$.



Now, since $\angle I_2O_1M$ is within and less than $\angle UO_1M$, it may be that T is exterior to $\angle I_2O_1M$. If this is the case, then we replace T by the point T' that is on c_1 and on the bisector of $\angle I_2O_1M$. Since $\angle T'O_1M < \angle TO_1M$ we can show, using Theorem 12.9 that this implies that $\overline{T'M} < \overline{TM}$ (exercise). Then, for T' we have $MT' < MT < \dfrac{t}{2}$.

We conclude that if $O_2M > r_2$, then for $t = O_2M - r_2$, there is a point T interior to $\angle I_2O_1M$ such that $MT < \dfrac{t}{2}$. By the triangle inequality, we have that $O_2M < MT + O_2T$. Since $MT < \dfrac{t}{2} < t$, then $O_2T > O_2M - MT > O_2M - \dfrac{t}{2} > O_2M - t = r_2$. Thus, T is outside c_2. But, this implies that M is between two points, R and T, that are both outside c_2. This contradicts Dedekind's property for rays.

A similar argument shows that $O_2M < r_2$ is impossible. Thus, $O_2M = r_2$ and M is on c_2.

To find the second intersection point, we can copy angle $\angle I_2O_1M$ to the angle on the other side of $\overleftrightarrow{O_1O_2}$, by congruence axiom III-4, yielding an angle $\angle I_2O_1M'$. By congruence axiom III-1 we can assume that $\overline{O_1M} \cong \overline{O_1M'}$. This implies that M' is on c_1. Then, $\Delta O_2O_1M \cong \Delta O_2O_1M'$ by SAS triangle congruence. Thus, $\overline{O_2M} \cong \overline{O_2M'}$, which means that M' is on c_2. □

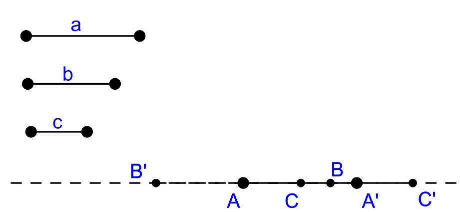
We can now complete the proof Euclid's Proposition 22, which we had to leave "dangling" in the last section.

Theorem 12.24. *Given three segments a, b, and c, if the sum of any two of these segments is always greater than the third, then there is a triangle with sides congruent to a, b, and c.*

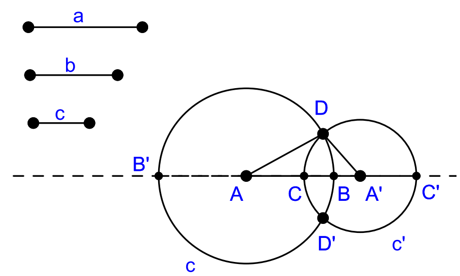
Proof: By repeated pair-wise comparison of the three segments, using segment ordering, we can assume that either 1) all segments are pair-wise non-congruent, 2) two of the three are congruent, or 3) all three are pair-wise congruent.

If all three segments are pair-wise non-congruent, then we can assume that $a > b > c$. Let l be a line and A a point on l. By Axiom III-1 in section 11.5 there is a point A' on l such that segment a is congruent to $\overline{AA'}$. Since $a > b$ then there is a point B between A and A' such that $\overline{AB}$ is congruent to b. Also, there is a point B' on the other side of A from B such that $\overline{AB'}$ is congruent to b. Since $a > c$ there is a point C between A and A' such that $\overline{A'C}$ is congruent to c. Also, there is a point C' on the other side of A' from C such that $\overline{A'C'}$ is congruent to c.

Since $b+c > a$, then $\overline{AB}+\overline{A'C} > \overline{AA'}$. Since B and C are between A and A', then, by exercise 11.8.4 we know that C is between A and B and B is between C and A'.



Construct the circle c centered at A of radius $\overline{AB}$ and the circle c' centered at A' of radius $\overline{A'C}$. Since C is on $\overline{AB}$, then $\overline{AC} < \overline{AB}$ and C is interior to circle c.



In Neutral geometry, lines do not "return" upon themselves, and so C' is not on $\overline{AB}$. Thus, $\overline{AC'} > \overline{AB}$ and C' is exterior to circle c. So, by the circle-circle continuity principle, circles c and c' must intersect at two points D and D'. Then, triangle $\Delta ADA'$ will have sides congruent to a, b, and c.

The case where two of a, b, and c are congruent can be argued in a similar fashion. The proof is left as an exercise.

The case where a, b, and c are all congruent is equivalent to the construction of an equilateral triangle, which was proven in Chapter!11.
□

This finishes our tour of Neutral geometry. The observant (and patient) reader will see that we have now proven all of the first 28 Propositions of Book I of *Elements*, plus Proposition 31. Euclid's Proposition 29 deals with the properties of a line crossing two parallel lines. The proof of this proposition relies on the uniqueness of parallels, and thus is outside of Neutral geometry. This is the first of Euclid's results that is uniquely a property of Euclidean geometry.

Exercise 12.2.1. *In the proof of Theorem 12.17 (Line-Circle Continuity) show that the point X given by Dedekind's axiom must be on $\overrightarrow{NP}$.*

Exercise 12.2.2. *In the proof of Theorem 12.17 (Line-Circle Continuity) show that there is a second intersection point of l with c on $\overrightarrow{NT}$.*

Exercise 12.2.3. *Finish the proof of Corollary 12.18. [Hint: In the proof of Theorem 12.17 we showed that the intersection points were on opposite rays $\overrightarrow{NS}$ and $\overrightarrow{NT}$, where N was interior to the circle and both of these rays were incident on l. There were two cases for N. Use these as the basis for your proof.]*

Exercise 12.2.4. *Prove Corollary 12.19. [Hint: See the hint for exercise 12.2.3. Also, make ample use of four-point betweenness.]*

Exercise 12.2.5. *Finish the proof of Theorem 12.24. That is, given three segments a, b, and c, where exactly two of the three are congruent, show there is a triangle with sides congruent to these three segments. [Hint: Follow the outline of the first part of the proof of Theorem 12.24, but use circles based on the two segments of differing lengths.]*

Exercise 12.2.6. *Let C be a circle and P a point in the plane. Show that P is either on the circle, outside the circle, or inside the circle.*