CHAPTER 15

Foundations of Projective Geometry

What a delightful thing this perspective is!

— Paolo Uccello (1379-1475) Italian Painter and Mathematician

15.1 AXIOMS OF PROJECTIVE GEOMETRY

In section 9.3 of Chapter 9 we covered the four basic axioms of Projective geometry:

P1 Given two distinct points there is a unique line incident on these points.

P2 Given two distinct lines, there is at least one point incident on these lines.

P3 There exist three non-collinear points.

P4 Every line has at least three distinct points incident on it.

We then introduced the notions of triangles and quadrangles and saw that there was a finite projective plane with 7 points and 7 lines that had a peculiar property in relation to quadrangles. The odd quadrangle behavior turns out to absent in most of the classical models of Projective geometry. If we want to rule out this behavior we need a fifth axiom:

Axiom P5: (Fano's Axiom) The three diagonal points of a complete quadrangle are not collinear.

This axiom is named for the Italian mathematician Gino Fano (1871–1952). Fano discovered the 7 point projective plane, which is now called the *Fano plane*. The Fano plane is the simplest figure which satisfies Axioms P1–P4, but which has a quadrangle with collinear diagonal points. (For more detail on this topic, review section 9.3.4.)

The basic set of four axioms is not strong enough to prove one of the classical theorems in Projective geometry —Desargues' Theorem.

Theorem 15.1. *Given two triangles ΔABC and $\Delta A'B'C'$, if the two triangles are perspective from a point O, then corresponding sides, when extended, intersect at three points which are collinear.*

The geometry of the Moulton Plane satisfies axioms P1–P4, but does not satisfy Desargues' Theorem. This was shown in exercises 9.3.11 and 9.3.12 in Chapter 9. If we want to guarantee that Desargues' Theorem holds, we need a new axiom:

Axiom P6: Given two triangles ΔABC and $\Delta A'B'C'$, if the two triangles are perspective from a point O, then corresponding sides, when extended, intersect at three points which are collinear.

The transformations of projective geometry that serve as the counterpart to the isometries of Euclidean and Non-Euclidean geometry are the *projectivities*. These are built of simple *perspectivities*. A persepectivity is defined in terms of *pencils.*

Definition 15.1. *The* pencil of points with axis l *is the set of all points on l. In a dual sense, the* pencil of lines with center O *is the set of all lines through point O. (Fig. 15.1)*

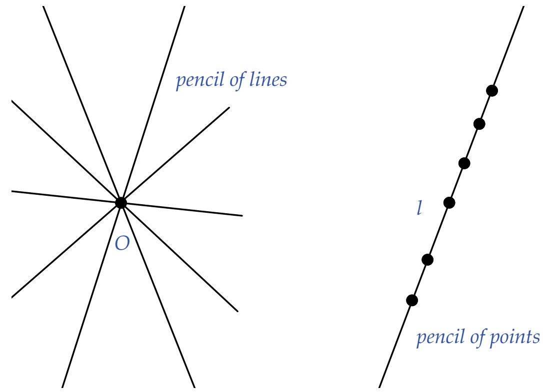


Figure 15.1

We define what the idea of being *perspective from O* in terms of pencils of points. We also note the dual definition for pencils of lines.

Definition 15.2. *A* perspectivity with center O *is a 1-1 mapping of a pencil of points with axis* l *to a pencil of points with axis* l' *such that if* A *on* l *is mapped to* A' *on* l'*, then* $\overleftrightarrow{AA'}$ *passes through* O*. In a dual sense, the* perspectivity with axis l *is a 1-1 mapping of a pencil of lines with center* O *to a pencil of points with center* O' *such that if line* a *through* O *is mapped to line* a' *through* O'*, then the intersection of* a *and* a' *lies on* l*.*

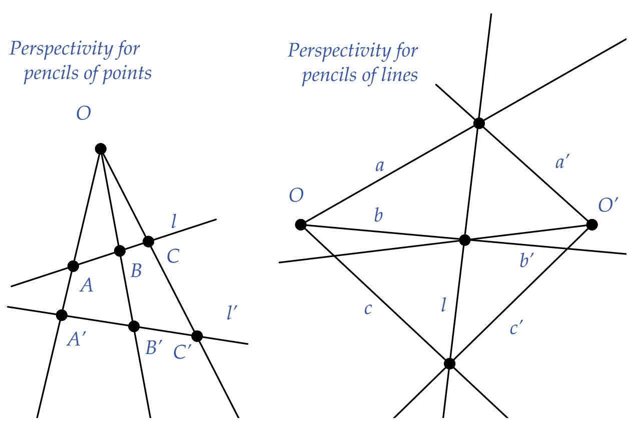


Figure 15.2

A perspectivity with center O is denoted by the symbol $\overset{O}{\barwedge}$. A perspectivity with axis l is denoted by the symbol $\overset{l}{\barwedge}$. Thus, in Fig. 15.2 we have $ABC \overset{O}{\barwedge} A'B'C'$ and $abc \overset{l}{\barwedge} a'b'c'$.

The composition of two perspectivities need not be a perspectivity. But, compositions of compositions of perspectivities will again be a composition of perspectivities. It is these compositions that form the *group* of transformations in Projective geometry. We call these transformations *projectivities.*

> **Definition 15.3.** *A mapping of one pencil into another is a* projectivity *if the mapping can be expressed as a finite composition of perspectivities.*

A projectivity is denoted by the symbol $\barwedge$. Thus, if $ABC \barwedge A'B'C'$ then there is a finite sequence of perspectivities that maps the collinear points ABC to the collinear points $A'B'C'$.

From Corollary 9.7, we know that there exist *non-identity* projectivities that map two points of a line back to themselves, i.e. that leave two points of a line invariant. This is quite different than what happens with isometries in Euclidean and Hyperbolic geometry. If an isometry fixes two points, then it is a reflection across the line through the two points, and it must leave every point on the line invariant. The isometry acts as the identity transformation on the line.

A natural question to ask, then, is whether a projectivity that fixes *three* points on a line must be the identity on that line. This result is provable in the Real Projective plane, but is not axiomatically provable from axioms P1-P4, or even if we add P5 and P6. If we want this property of projectivities to be true in Projective geometry, we must add this property as an additional *axiom.*

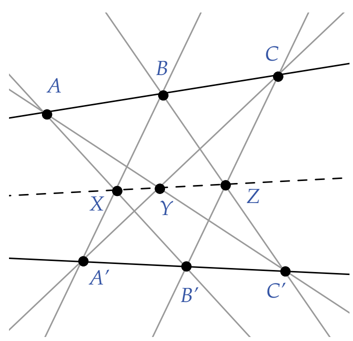
Axiom P7: If a projectivity leaves three distinct points on a line invariant, then the projectivity must be the identity on the pencil of points on that line.

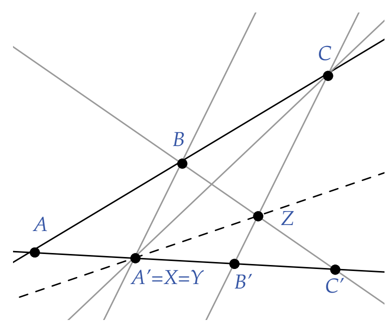
We will now prove that this axiom is logically equivalent to Pappus's Theorem. We will use the a shorthand notation for the intersection point of two lines or the line defined by two points.

Definition 15.4. *Given two distinct lines a and b, $a \cdot b$ is the unique point of intersection of these lines. Given two distinct points A and B, AB is the unique line defined by these two points.*

Theorem 15.2. *(Pappus's Theorem) Let A,B,C be three distinct points on line l and A', B', and C' be three distinct points on line l', with $l \neq l'$. Then, the intersection points $X = AB' \cdot A'B$, $Y = AC' \cdot A'C$, and $Z = BC' \cdot B'C$ are collinear.*

There are two possible configurations of points in Pappus's theorem. In the one at the left, none of the points on the two lines are the point of intersection of the lines. In the figure on the right, one of the points, A, is the intersection point of the lines. Pappus's Theorem can be proven in this case, using results solely based on axioms P1-P4. This was shown in exercise 9.4.4.



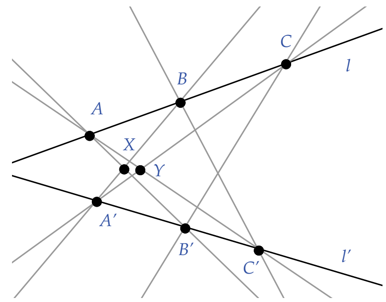


In order to show that Pappus's Theorem and axiom P7 are *logically equivalent*, we will to carry out two proofs. We start by showing that axioms P1-P4 and P7 imply Pappus's Theorem.

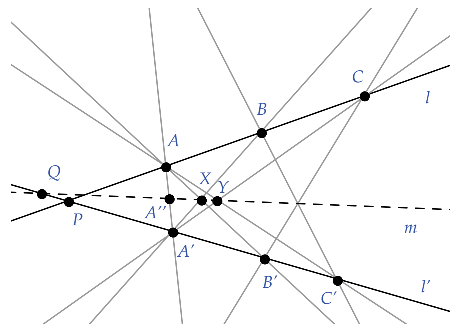
Theorem 15.3. *If one assumes axioms P1-P4 and axiom P7, then Pappus's Theorem is true.*

Proof: We start with the assumption of Pappus's Theorem. Let A, B, C be three distinct points on line l and A', B', and C' be three distinct points on line l', with $l \neq l'$. By the comments above concerning the two possible configurations of points in Pappus's theorem, we can assume that none of A,B, or C are on l' and none of A', B', or C' are on l.

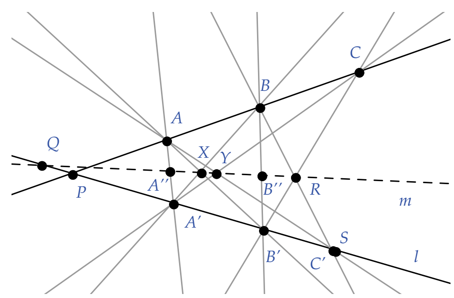
Let $X = AB' \cdot A'B$ and $Y = AC' \cdot A'C$. Then, neither of X nor Y is on l or l'.



Let $m = XY$, and let A'' be the intersection of m with AA'. Then, $ABC \overset{A'}{\barwedge} A''XY \overset{A}{\barwedge} A'B'C'$. Thus, there is a projectivity mapping ABC to $A'B'C'$. Let $P = l \cdot l'$ and $Q = l' \cdot m$. Then $ABP \overset{A'}{\barwedge} A''XQ \overset{A}{\barwedge} A'B'Q$.



Now, let $R = m \cdot B'C$, $S = l' \cdot BR$ and $B'' = m \cdot BB'$. Then, $ABP \overset{B'}{\barwedge} XB''Q \overset{B}{\barwedge} A'B'Q$. So, we have two projectivities that both take ABP to $A'B'Q$. It follows from Axiom P7 (exercise 9.4.5) that the two projectivities must be the same on l.



Consider point C. Under the first projectivity (from A' and then A) C goes to Y and then to C'. Under the second projectivity (from B' and then from B) C goes to R and then to S. Thus, $S = C'$ and $\overleftrightarrow{BR} = \overleftrightarrow{BC'}$. Since R is on m, then $R = BC' \cdot B'C$ is collinear with X and Y. $\square$

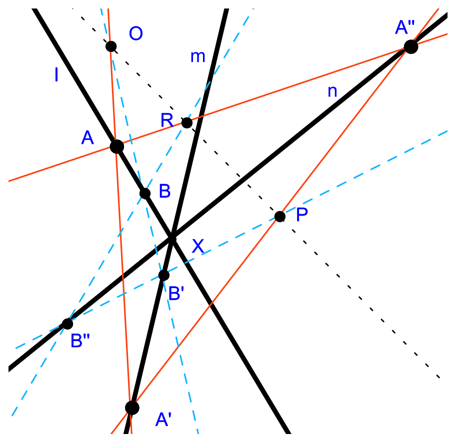
To show that axioms P1-P4 and Pappus's Theorem imply axiom P7, one also needs to assume axiom P6, Desargues Theorem. (In the exercises for section 9.4, we proved that P6 can be proven from Pappus's Theorem, so this is not that much of an assumption.)

We first prove a series of lemmas showing that we can reduce the number of perspectivities that make up a projectivity. Our exposition in this section follows closely that of Hartshorne [12, Chapter 5]

Lemma 15.4. *Let l, m, and n be three lines. Suppose there is a projectivity $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$. If the lines are* concurrent *at point X, then there is a point R on $\overleftrightarrow{OP}$ such that the perspectivity $l \overset{R}{\barwedge} n$ exactly matches the original projectivity.*

Proof:

With the given assumptions let A and B be two distinct points on l. Let $A \overset{O}{\barwedge} A' \overset{P}{\barwedge} A''$, with A' on m and A'' on n. Likewise, let $B \overset{O}{\barwedge} B' \overset{P}{\barwedge} B''$. Then, triangles $AA'A''$ and $BB'B''$ are perspective from X. Thus, by Desargues Theorem we have that $AA'.BB' = O$, $A'A''.B'B'' = P$, and $R = AA''.BB''$ are collinear. (Note that this argument works even if $l = n$.)



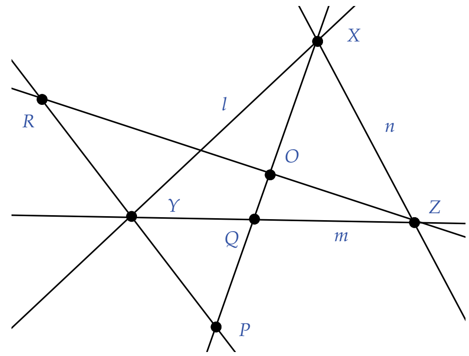
Let C be a point distinct from A and B on l and let $C \overset{O}{\barwedge} C' \overset{P}{\barwedge} C''$. Then, by the reasoning above, we have that $S = AA''.CC''$ and $T = BB''.CC''$ are both on $\overleftrightarrow{OP}$. We conclude that the three lines AA'', BB'', and CC'' all intersect along the same line. Then, it must be the case that, either at least two of these lines are the same line, or they are concurrent at a single point (exercise).

No pair of the three lines can lie on the same line, for if they did, then O would be on l. We conclude that $R = S = T$. Thus, the perspectivity from R matches the projectivity, as C can be chosen arbitrarily on l. □

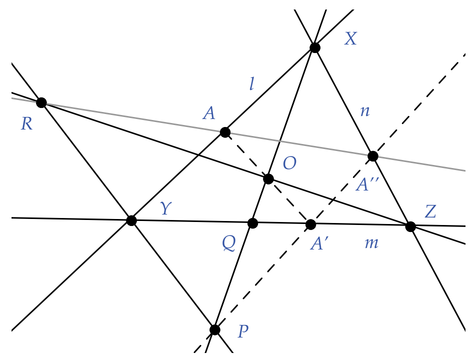
Lemma 15.5. *Let l, m, and n be three lines, with $l \neq n$. Suppose there is a projectivity $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$. If the lines are not concurrent, and $X = l \cdot n$ is invariant under the projectivity, then l is perspective to n. That is, there is a point R such that $l \overset{R}{\barwedge} n$ exactly matches the original projectivity.*

Proof:

Let $X = l \cdot n$, $Y = l \cdot m$, and $Z = m \cdot n$. Also, let $Q = m \cdot PX$ and $R = PY \cdot OZ$. Under the perspectivity from O, X goes to Q. Since X must be invariant, then $Q \overset{P}{\barwedge} X$, which implies that O,P, and X must be collinear.



Let A be any point on l and let $A \overset{O}{\barwedge} A'$, $A' \overset{P}{\barwedge} A''$. Then, Y, A', and Z are three distinct points on m, while O, X, and P are three distinct points on XP. Applying Pappus's Theorem to these two triples, we get that $A = YX \cdot OA'$, $R = YP \cdot OZ$, and $A'' = A'P \cdot XZ$ are collinear.



Now, R is defined independent of the point A. So, for any A on l we get that the line through R and A will yield the point A'', which is the result of the projectivity applied to A. Thus, the projectivity is the same as the perspectivity from R. □

Lemma 15.6. *Let l, m, and n be three lines with $l \neq n$. Suppose there is a projectivity $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$ and let $X = l \cdot n$. If the lines are not concurrent, and $X = l \cdot n$ is not invariant under the projectivity, then there is a line m' and points O' on n and P' on l such that $l \overset{O'}{\barwedge} m' \overset{P'}{\barwedge} n$ exactly matches the original projectivity.*

Proof: If O is on n, then we choose $O' = O$ and $m = m'$. So, we assume O is not on n. We know that O, P, and X are not collinear, for if they were collinear, then X would be invariant under the projectivity. Let $O' = OP \cdot n$. Then, $O' \neq X$. Let $A \neq X$ and $B \neq X$ be points on l with $AB \overset{O}{\barwedge} A'B' \overset{P}{\barwedge} A''B''$. Let $A^* = O'A \cdot PA''$ and $B^* = O'B \cdot PB''$ (Fig. 15.3).

We know that $A^* \neq B^*$, as if $A^* = B^*$, then A and B would be on a line containing O', which is impossible.

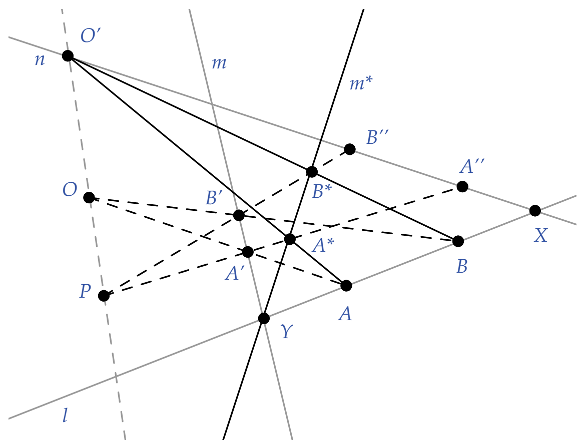


Figure 15.3

Consider the perspectivity from O' on l. If $A = A'$, then the perspectivity from O' maps A to itself and then the perspectivity from P maps $A' = A$ to A''. Thus, $A \overset{O'}{\barwedge} A' \overset{P}{\barwedge} A''$.

So, we assume that $A \neq A'$. We will show that neither A' nor A can equal A^*. Suppose that $A^* = A$. Then, PA' and OA' must be the same line, and so $P = O$. But, if $P = O$, then X is invariant under the projectivity. So, $A^* \neq A$. Suppose $A^* = A'$. Then, $O'A$ and OA are the same line and $O = O'$. But, we are assuming that O is not on n. Thus, $A^* \neq A$.

So, $AA'A^*$ is a triangle. Let $B \neq A \neq X$ be another point on l and let $B \overset{O}{\barwedge} B' \overset{P}{\barwedge} B''$. By the reasoning above $BB'B^*$ is a triangle.

Both of these triangles are perspective from line OP. Thus, by Desargues Theorem, they must be perspective from a point. It is clear that this is the point $Y = m.l$. Let $m^* = A^*B^*$ and let C be any other point on l that is not X. Then, $CC'C^*$ will again be a triangle perspective from Y in comparison with $AA'A^*$. So, C^* will be on m.

Then, $l \overset{O'}{\barwedge} m^* \overset{P}{\barwedge} n$ matches the original projectivity everywhere but at X. It will be left as an exercise to show that we can make our argument work for the point X.

A similar argument can be used to find P' and m' such that $l \overset{O'}{\barwedge} m' \overset{P'}{\barwedge} n$ matches $l \overset{O'}{\barwedge} m^* \overset{P}{\barwedge} n$. $\square$

The next result finishes our analysis of a projectivity built from two perspectivities.

Lemma 15.7. *Let l, m, and n be three lines with $l \neq n$. Suppose there is a projectivity $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$ and let $X = l \cdot n$. Let $m' \neq l$ be a line distinct from m that passes through $Y = l \cdot m$. If l, m, and n are not concurrent and $X = l \cdot n$ is not invariant under the projectivity, then there is a point O' on OP such that $l \overset{O'}{\barwedge} m' \overset{P}{\barwedge} n$ exactly matches the original projectivity.*

Proof: As in the previous lemma, we know that O, P, and X are not collinear. Let $A \neq Y$ be a point on l and let $A \overset{O}{\barwedge} A' \overset{P}{\barwedge} A''$ for points A' on m and A'' on n. Let $A^* = PA''.m'$ (Fig. 15.4).

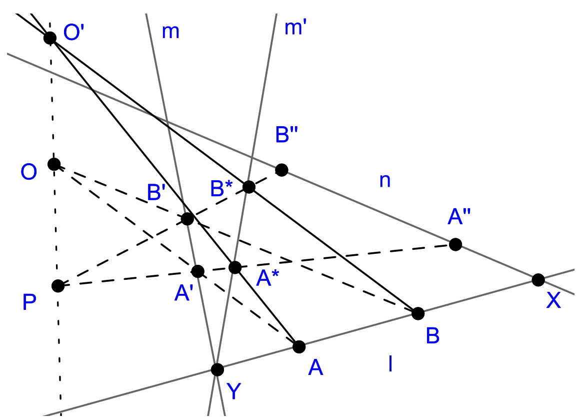


Figure 15.4

Since m' is not m or l, we have that $A \neq A^*$ and $A' \neq A^*$. Thus, $AA'A^*$ is a triangle (even if $A = X$). Let $B \neq A \neq Y$ be another point on l. Then, $BB'B^*$ is a triangle. These two triangles are perspective from Y. Thus, by Desargues Theorem, they are perspective from a line, which must be OP. We conclude that $O' = AA^*.BB^*$ is on OP. Let C be any other point on l other than Y. Then, $CC'C^*$ is a triangle perspective from Y. By the argument just given, $AA^*.CC^*$ must be on OP, and so $AA^*.CC^* = O'$.

Since C can be chosen arbitrarily, we conclude that $l \overset{O'}{\barwedge} m' \overset{P}{\barwedge} n$ matches $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$ everywhere, except possibly at Y. However, the perspectivity from O' will fix Y, so $l \overset{O'}{\barwedge} m' \overset{P}{\barwedge} n$ matches $l \overset{O}{\barwedge} m \overset{P}{\barwedge} n$ at Y. □

Note that this theorem has a symmetric counterpart. We could have assumed that $m' \neq l$ was a line distinct from m that passes through $V = n \cdot m$. Then, if l, m, and n are not concurrent and $X = l \cdot n$ is not invariant under the projectivity, then there must be a point P' on OP such that $l \overset{O}{\barwedge} m' \overset{P'}{\barwedge} n$ exactly matches the original projectivity.

We can now prove that a projectivity is essentially two perspectivities.

Theorem 15.8. *A projectivity between two distinct lines can be written as the composition of at most two perspectivities.*

Proof: A projectivity is defined as the composition of a finite number of perspectivities. Thus, it is enough to show that a projectivity that is composed of three perspectivities can be written as the composition of two, since we can then repeatedly reduce the original number of perspectivities to at most two. (The original might be a single perspectivity)

We assume, then, that the given projectivity is $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$, with $l \neq o$. If $l = n$, then $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n$ can be replaced by a single perspectivity, by Lemma 15.4. Similarly, if $m = o$, then $m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$ can be replaced by a single perspectivity. The definition of a perspectivity implicitly assumes that the two lines under the perspectivity are distinct, thus $l \neq m$, $m \neq n$, and $n \neq o$. We conclude that we can assume that l, m, n, and o are all distinct.

If l, m, n or m, n, o are concurrent, then we can reduce the three perspectivities to two using Lemma 15.4. If $L = l \cdot n$ is invariant under $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n$, or if $M = m \cdot o$ is invariant under $m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$, then we can use Lemma 15.5 to reduce the three perspectivities to two. Thus, we can assume that l, m, n and m, n, o are not concurrent and that the points L and M defined above are not invariant under the perspectivities $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n$ and $m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$.

Now consider the case where l, m, o are concurrent. Let $Y = l \cdot m$ and $V = n \cdot m$. By the dual to axiom P3, there must be some line m'

through V that is different than o and that does not pass through Y. By the note after Lemma 15.7, we know that there is a point Q' of PQ such that $l \overset{P}{\barwedge} m' \overset{Q'}{\barwedge} n$ which preserves the original perspectivity from l to n. So, we can equivalently consider the projectivity $l \overset{P}{\barwedge} m' \overset{Q'}{\barwedge} n \overset{R}{\barwedge} o$, where l, m', o are not concurrent.

If l, n, and o are concurrent, we could similarly find a line n' through $U = n \cdot o$ that misses $T = l \cdot n$ and point R' on QR such that $l \overset{P}{\barwedge} m' \overset{Q'}{\barwedge} n \overset{R'}{\barwedge} o$ matches $l \overset{P}{\barwedge} m' \overset{Q'}{\barwedge} n' \overset{R}{\barwedge} o$.

So, we can assume that we have a projectivity $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$ with none of $\{l, m, n\}$, $\{m, n, o\}$, $\{l, m, o\}$, nor $\{l, n, o\}$ concurrent. Also, we can assume that neither of the two intersection points $L = l \cdot n$ or $M = m \cdot o$ are invariant under their respective perspectivities, as if they were, then we can use Lemma 15.5 to reduce the three perspectivities to two.

Using Lemma 15.6, we can assume that Q is on line o. This would perhaps require a change in n. A quick look at the proof of the lemma shows that this change in n does not affect point $T = l \cdot n$. Thus, it does not change the non-concurrency of l, n, o. A change in n might affect the assumptions about l, m, n and m, n, o being not concurrent, or it might make $L = l \cdot n$ invariant. However, if either l, m, n or m, n, o are concurrent, we could use Lemma 15.4 to reduce the three perspectivities to two. Also, if $L = l \cdot n$ becomes invariant, we can use Lemma 15.5 to reduce the three perspectivities to two.

So, we can assume that we have a projectivity $l \overset{P}{\barwedge} m \overset{Q}{\barwedge} n \overset{R}{\barwedge} o$ with none of $\{l, m, n\}$, $\{m, n, o\}$, $\{l, m, o\}$, nor $\{l, n, o\}$ concurrent. Also, we can assume that Q is on line o. Let $Z = n \cdot o$ and $h = YZ$ (Fig. 15.5).

Since Y is not on o, then Q is not on h. Let A and B be on l and let $AB \overset{P}{\barwedge} A'B' \overset{Q}{\barwedge} A''B'' \overset{R}{\barwedge} A'''B'''$. Since m, n, and o are not concurrent, then h cannot be m or n. Thus, there is a perspectivity from Q mapping m to h. Let H and J be on h such that $A'B' \overset{Q}{\barwedge} HJ$.

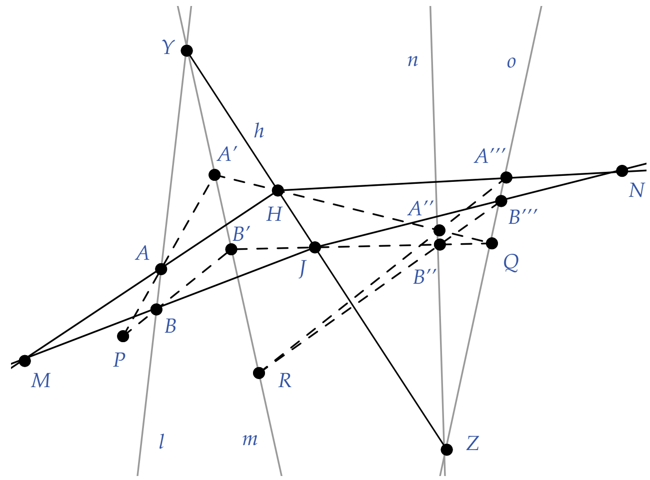


Figure 15.5

Now, $h \neq o$, as l, m, and o are not concurrent. Likewise, $h \neq l$, as l, n, and o are not concurrent. Thus, $HA''A'''$ and $JB''B'''$ are triangles, as are $AA'H$ and $BB'J$. We see that $HA''A'''$ and $JB''B'''$ are perspective from Z. Thus, by Desargues Theorem, they are perspective from a line, which must be QR. Thus, $N = HA''' \cdot JB'''$ is on QR. Similarly, $AA'H$ and $BB'J$ are perspective from Y and so $M = HA \cdot JB$ is on QP. Then, $AB \overset{M}{\barwedge} HJ \overset{N}{\barwedge} A'''B'''$. If C is any other point on l, we would likewise get $C \overset{M}{\barwedge} K \overset{N}{\barwedge} C'''$ for some K on h, because M has to be the point on QR intersected by HA and N is the point on QR intersected by HA''' and so are independent of C. We conclude that $l \overset{M}{\barwedge} h \overset{N}{\barwedge} o$ matches the original projectivity. □

We can now complete the demonstration of the equivalence of Pappus's Theorem with Axiom P7 on projectivities.

Theorem 15.9. *If we assume Axioms P1-P4 and Pappus's Theorem, then Axiom P7 holds. That is, a projectivity that leaves three distinct points on a line invariant must be the identity.*

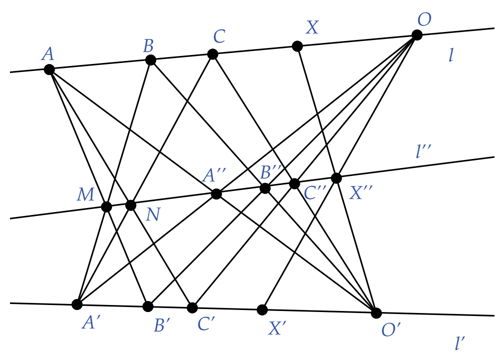
Proof: Let π be the projectivity. Assume the projectivity leaves points A, B, and C on line l invariant. There are two cases to consider —either l has exactly three points or it has more than three points.

Case I: If l has exactly three points, then every line has exactly three points (exercise). By the dual to axiom P4, there is a line $l' \neq l$. Let A', B', and C' be points on l. We know that l and l' intersect. We can assume they intersect at $A = A'$. Let $R = BB' \cdot CC'$. Then, R is not on l or l', and there is a perspectivity from R taking B and C to B' and C'. Let ρ be the perspectivity. Now, consider the composition of $\rho \circ \pi$. This gives another projectivity mapping B and C to B' and C'. But if every line has exactly three points, there is only one projectivity possible from l to l' mapping B and C to B' and C' (exercise). Thus, $\rho \circ \pi = \rho$. Since a perspectivity is invertible, we get that π equals the identity.

Case II: If l has at least four points, say A, B, C, and D, then by the dual to axiom P4, there is a line l' through D that is not l, and thus does not go through A, B, or C. We know by axiom P4 that there are at least two other points A' and B' on l' distinct from D. Let $R = AA' \cdot BB'$. Then, R is not on l or l', and there is a perspectivity from R taking A and B to A' and B'. This perspectivity maps C to some point C' on l' other than A' or B'. Also, $C' \neq D$, as R is not on l. We have thus defined a projectivity taking A, B, C to A', B', C'. We claim that this projectivity is unique. Suppose there was a second projectivity mapping A, B, and C to A', B', and C'. If this second projectivity is a perspectivity, then it would have to be the perspectivity from R.

If the second projectivity is made up of two or more perspectivities, then we know by the preceding theorem that it can be written as the composition of two perspectivities, say $l \overset{O'}{\barwedge} l'' \overset{O}{\barwedge} l'$. If $D = l \cdot l'$ is invariant under the projectivity $l \overset{O'}{\barwedge} l'' \overset{O}{\barwedge} l'$, then by Lemma 15.5, we know that the projectivity is the same as a perspectivity, and thus must be the perspectivity from R.

So, we can assume that D is not invariant under the projectivity, and by Lemma 15.6, we can assume that O is on l and O' is on l'. We have $ABC \overset{O'}{\barwedge} A''B''C'' \overset{O}{\barwedge} A'B'C'$. Applying Pappus's Theorem to A, B, O on l and A', B', O' on l', we have $M = AB' \cdot A'B$, $A'' = AO' \cdot A'O$, and $B'' = BO' \cdot B'O$ are collinear. Thus, M is on $l'' = A''B''$. Likewise, $N = AC' \cdot A'C$ is on l''.



In the proof of Theorem 9.6, the line through M and N was exactly the line used for the construction of the projectivity $l \overset{A'}{\barwedge} l'' \overset{A}{\barwedge} l'$. All that is left to show is that every point X on l that goes to the point X' on l' via $l \overset{O'}{\barwedge} l'' \overset{O}{\barwedge} l'$ will also go to X' by $l \overset{A'}{\barwedge} l'' \overset{A}{\barwedge} l'$. Applying Pappus's Theorem to AXO and $A'X'O'$ we get that A'', X'', and $AX' \cdot A'C$ are collinear, so $AX' \cdot A'C$ is on l'', and thus $l \overset{A'}{\barwedge} l'' \overset{A}{\barwedge} l'$ maps X to X'.

Now, $l'' = MN$ with $M = AB' \cdot A'B$ and $N = AC' \cdot A'C$. So, we have shown that every projectivity mapping A, B, and C to A', B', and C' is equivalent to $l \overset{A'}{\barwedge} l'' \overset{A}{\barwedge} l'$, which only depends on A, B, C, A', B', and C'. Thus, there is a unique projectivity taking A, B, C on l to A', B', C' on l'. The rest of the proof follows much like it did in Case I where all lines had three points and will be left as an exercise. □

Exercise 15.1.1. *Show that the set of projectivities of a line l into itself forms a group. Refer to Exercise 5.6.5 for the definition of a group.*

Exercise 15.1.2. *Prove that if one assumes axioms P1-P4, and Papus's Theorem, then if two projectivities between pencils of lines with centers O and O' both have the same values on three distinct lines a, b, and c through O, then the projectivities must be the same on the pencil of lines at O.*

Exercise 15.1.3. *Suppose three lines all intersect along the same line. Prove that either at least two of the lines are the same line or they are concurrent at a single point.*

Exercise 15.1.4. *Finish the proof of Lemma 15.6. That is, show that the argument given at the end of the proof works for the point $X = l \cdot n$.*

Exercise 15.1.5. *Assuming axioms P1-P4, show that if there exists a line with exactly three points, then every line has exactly three points. [Hint: Suppose there was another line with four points. Define a perspectivity to reach a contradiction.]*

Exercise 15.1.6. *Assume axioms P1-P4 and assume that every line has exactly three points. Let l and l' be distinct lines with A, B, and C on l and A', B', and C' on l'. Show that there is only one projectivity possible from l to l' that takes A, B, and C to A', B', and C'.*

Exercise 15.1.7. *Finish the last part of the proof of Theorem 15.9. That is, show that if there is a unique projectivity mapping distinct points A, B, and C on line l to distinct points A', B', C' on line $l' \neq l$, then a projectivity that leaves three distinct points on a line invariant must be the identity.*

15.2 HOMOGENEOUS COORDINATES AND TRANSFORMATIONS IN THE REAL PROJECTIVE PLANE

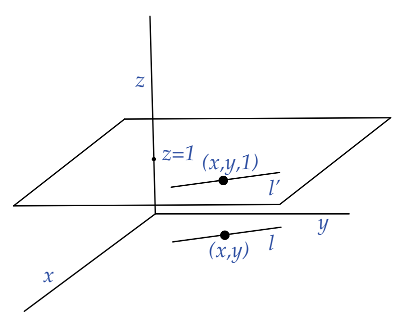
To define the Real Projective plane, we need the definition of points at infinity and the line at infinity.

Definition 15.5. *Let l be a Euclidean line and let $[l]$ represent the set of all Euclidean lines that are parallel to l. Then, we define $[l]$ to be an* ideal point *or a* point at infinity. *The set of all ideal points is called the* line at infinity. *Given a Euclidean line l, we define the* extended line *through l to be the set consisting of the points of l plus $[l]$.*

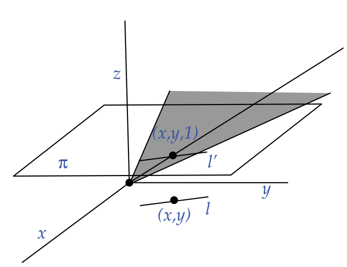
The Real Projective plane consists of all ordinary Euclidean points plus all ideal points. Real projective lines consist of all extended Euclidean lines plus the line at infinity.

In the three dimensional model of the Real Projective Plane, we interpret the two dimensional points of the Euclidean plane as three dimensional points with z-coordinate equal to 1.

Given a Euclidean point (x, y) we identify this point with the point $(x, y, 1)$. An ordinary Euclidean line will then be identified with the corresponding line at height 1.



There is a 1–1 relationship between points $(x, y, 1)$ and lines passing through the origin and these points. Also, there is a similar relationship between lines l' in the $z = 1$ plane (labeled π in the figure) and *planes* through the origin and l'.



We can identify the ordinary points of the Real Projective Plane with points $(x, y, 1)$, or equivalently with any point along the line $k(x, y, 1)$. For ideal points, as a point moves farther and farther from the z-axis, the slope of the line through the origin and that point will decrease. In the "limit," this line will approach a line in the x-y plane. Thus, points at infinity can be identified with lines through the origin with z-coordinate equal to zero.

We can use *homogeneous coordinates* to analytically capture properties of points and lines in the Real Projective plane. .

Definition 15.6. Homogeneous coordinates (x_1, x_2, x_3) *for an ordinary Euclidean point* $P = (x, y)$ *are a choice of* x_1, x_2, x_3 *such that* $\frac{x_1}{x_3} = x$ *and* $\frac{x_2}{x_3} = y$. *One such representation is* $(x, y, 1)$. *Homogeneous coordinates of the form* $(x_1, x_2, 0)$ *represent points at infinity.*

The 3-D model of the Real Projective Plane uses this homogeneous representation of points and lines.

Definition 15.7. *The points of the 3-D model include all non-zero homogeneous coordinate vectors* $v = (x, y, z)$. *The lines of the 3-D model include all non-zero homogeneous planar vectors* $[a, b, c]$. *A point* $v = (x, y, z)$ *is on a line* $u = [a, b, c]$ *iff* $ax + by + cz = 0$. *That is, if the dot product* $u \cdot v = 0$.

For the 3-D model we will think of vectors for lines $[a, b, c]$ as row vectors and vectors for points (x, y, z) as column vectors. Then, $u \cdot v$ can be thought of as a matrix multiplication.

15.2.1 Transformations

In section 9.5 we used homogeneous coordinates to parameterize points on a line. We proved the following:

Theorem 15.10. *Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be two distinct points with homogeneous coordinates in the Real Projective Plane. Let $X = (x_1, x_2, x_3)$ be any point on the line through P and Q. Then, X can be represented as $X = \alpha P + \beta Q = (\alpha p_1 + \beta q_1, \alpha p_2 + \beta q_2, \alpha p_3 + \beta q_3)$, with α and β real constants, with at least one being non-zero. Conversely, if $X = \alpha P + \beta Q$, with at least one of α and β not zero, then X is on the line through P and Q.*

Definition 15.8. *The coordinates (α, β) are called* homogeneous parameters *or* parametric homogeneous coordinates *of the point X with respect to the* base points *P and Q.*

Homogeneous parameters of a point are not unique. Any multiple of a given homogeneous representation, say (α, β) is equivalent to any non-zero scalar multiple of that representation $(\lambda\alpha, \lambda\beta)$. What is unique is the *ratio* of the two parameters $\frac{\alpha}{\beta}$.

Definition 15.9. *Given homogeneous parameters (α, β) of the point X with respect to the* base points *P and Q, the ratio $\frac{\alpha}{\beta}$ is called the* parameter *of the point.*

A projectivity has a homogeneous coordinate representation.

Theorem 15.11. *Given a projectivity $l \,\overline{\wedge}\, l'$, with l and l' distinct. Let P and Q be points on l with $PQ \overline{\wedge} P'Q'$. Let X be on l with coordinates (α, β) with respect to P and Q. Let X' be the point on l' with $PQX \overline{\wedge} P'Q'X'$, and let X' have coordinates (α', β') with respect to points P' and Q'. Then, the projectivity, as a mapping of points on l, can be represented by a matrix equation of the form:*

$$\lambda \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where the determinant $ad - bc \neq 0$ and $\lambda \neq 0$.

The converse to the above result also holds – every non-singular $2x2$ matrix generates a projectivity with respect to local homogeneous coordinates.

Theorem 15.12. *Let P and Q be distinct points on l and let P', Q' distinct points on l'. Let X be a point on l with parametric homogeneous coordinates (α, β) with respect to P and Q, and let the homogeneous coordinates of a point X' on l' be (α', β'). Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with $ad - bc \neq 0$. Then, the transformation of l into l' defined by the matrix equation:*

$$\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

defines a projectivity *from l to l'.*

The proofs of these theorems can be found in section 9.5.

A projectivity maps a line to another line. A collineation is a map of the entire projective plane to itself.

Definition 15.10. *A* collineation *is a 1-1 and onto transformation of the Real Projective Plane to itself that maps lines to lines and preserves intersections of lines.*

Theorem 15.13. *A collineation can be represented by a matrix equation of the form $\lambda X' = AX$ where A is a $3x3$ matrix with non-zero determinant and $\lambda \neq 0$.*

Collineations were shown to have an interesting connection with complete quadrangles. A complete quadrangle consists of four distinct points, no three of which are collinear.

Theorem 15.14. *There exists a unique collineation that takes the four points of a complete quadrangle to any other four points of another complete quadrangle.*

While the proof of the Fundamental Theorem of Projective Geometry presented in section 9.5 is based on using homogeneous coordinates in the Real Projective Plane, the result actually holds for any abstract Projective geometry that satisfies Fano's axiom and Pappus's Theorem. A proof of this result can be found in [11][Chapter 8].

So far in this chapter, perspectivities can be maps from a pencil of points on one line to a pencil of points on another, or from a pencil of lines on one point to a pencil of points on another (Fig. 15.6.)

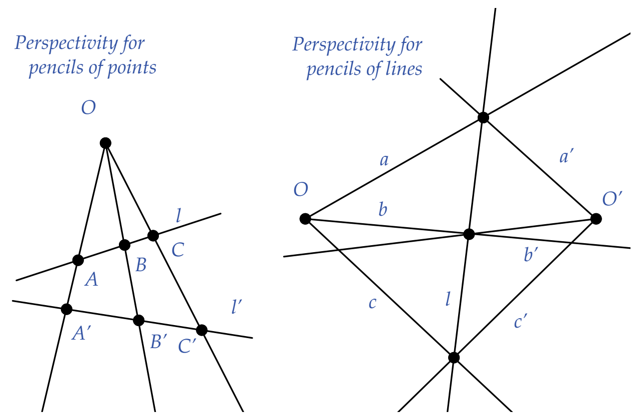
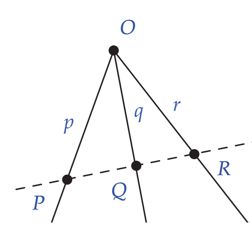


Figure 15.6

There is a third type of perspectivity —that of a pencil of points to a pencil of lines, or vice-versa.

Definition 15.11. *A* perspectivity with center O and axis l *is a 1-1 mapping of a pencil of points with axis* l *to a pencil of lines with center* O *such that if* P *on* l *is mapped to* p *through* O*, then* p *passes through* P.

Here we have an illustration of a perspectivity of this third type. Lines p, q, and r are mapped to points P, Q, and R.

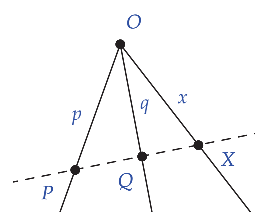


If we are careful with the choice of homogeneous coordinates for lines and points, we can synchronize the coordinates for a perspectivity from a pencil of points to a pencil of lines.

Theorem 15.15. *Given a perspectivity with center O and axis l, we can choose homogeneous coordinates for the pencil of points on l and the pencil of lines through O so that the same coordinates can represent both sets.*

Proof:

Let P and Q be points on l and let p and q be the corresponding lines through O. Choose P and Q as the base points for homogeneous coordinates on l and choose p and q as base lines for homogeneous coordinates for lines through O.



Then, the coordinates $(1, 0)$ will represent P on l and will also represent p for the pencil of lines at O. Likewise, $(0, 1)$ will represent Q and q. Consider the line $x = \alpha p + \beta q$. We will show that the point with coordinates $X = \alpha P + \beta Q$ on l is also on x.

We know that the vector p represents the normal vector to the plane through the origin containing vectors P and O. Thus, $p \cdot P = 0$ and p can be chosen as $P \times O$. Likewise, $q \cdot Q = 0$ and $q = Q \times O$. We need to show that $x \cdot X = 0$.

We have

$$\begin{aligned} x \cdot X &= (\alpha p + \beta q) \cdot (\alpha P + \beta Q) \\ &= \alpha^2 (p \cdot P) + \alpha\beta(p \cdot Q + q \cdot P) + \beta^2 (q \cdot Q) \\ &= \alpha\beta(p \cdot Q + q \cdot P) \\ &= \alpha\beta[(P \times O) \cdot Q) + (Q \times O) \cdot P] \end{aligned}$$

Now, $(U \times V) \cdot W$ is the triple product of the vectors U, V, and W. By the properties of the triple product, if we reverse the first and last vectors, the triple product reverses sign. Thus, we get $x \cdot X = \alpha\beta(P \times O) \cdot Q - (P \times O) \cdot Q = 0$.

□

Exercise 15.2.1. *Let a collineation be represented by the 3x3 matrix A. Let $u = [u_1, u_2, u_3]$ be the coordinate representation of a line. Show that the image of u under the collineation has coordinates $ku' = uA^{-1}$, where A^{-1} is the inverse to A. [Hint: Let line u have equation $uX = 0$ and u' have equation $u'X' = 0$. Note that X goes to $sX' = AX$.]*

Exercise 15.2.2. *Let* $P = (1,3,1)$, $Q = (0,1,1)$, $R = (3,0,1)$, *and* $S = (4,2,1)$. *Show that* $PQRS$ *is a complete quadrangle.*

Exercise 15.2.3. *Show that a collineation has at least one invariant point and one invariant line. [Hint: Consider the eigenvalues for the matrix representing the collineation.]*

Definition 15.12. *A collineation is called a* perspective collineation *if there exists a unique line for which every point on the line is fixed by the collineation. This line is called the* axis *of the collineation.*

Exercise 15.2.4. *Show that a perspective collineation can have at most one invariant point that is not on its axis. [Hint: Use Theorem 15.14.]*

Exercise 15.2.5. *Given a perspective collineation, prove that there is a unique point* C *with the property that every line through* C *is invariant under the collineation. [Hint: There are two cases: either an invariant point exists off the axis, or invariant points are only on the axis. In the second case, let* P *be a point not on the axis* (l) *and let* P' *be the image of* P *under the collineation. Let* $m = PP'$ *and let* $C = l \cdot m$. *Show that* C *is the desired point.]*

Definition 15.13. *Given a perspective collineation, the point* C *from Exercise 15.2.5 is called the* center *of the collineation.*

Definition 15.14. *A non-identity perspective collineation is called an* elation *if its center lies on its axis. The collineation is called a* homology *if its center does not lie on its axis.*

Exercise 15.2.6. *Show that every Euclidean reflection is a homology, when considered as a collineation in the Real Projective Plane (the Euclidean plane plus all points at infinity).*

Exercise 15.2.7. *Show that every Euclidean translation (with nonzero direction vector) is an elation, when considered as a collineation in the Real Projective Plane (the Euclidean plane plus all points at infinity).*

Exercise 15.2.8. *Show that a Euclidean rotation, that is not the identity or a half-turn, is not a perspective collineation, when considered as a collineation in the Real Projective Plane (the Euclidean plane plus all points at infinity).*

15.3 PROJECT 21 - INTRODUCTION TO CONICS

In this Project we will see that the analog of circles in Projective geometry is the idea of *conic sections*. These include all of the traditional sections of the cone – the circle, ellipse, hyperbola, and parabola.

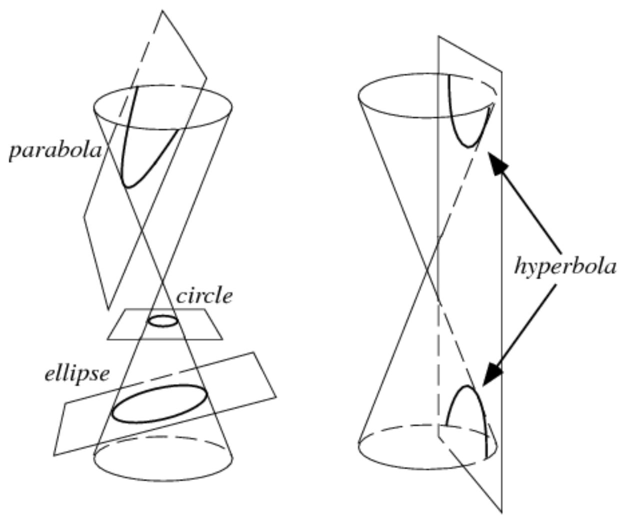


Figure 15.7 Conic Sections - from *MathWorld*–A Wolfram Web Resource [18]

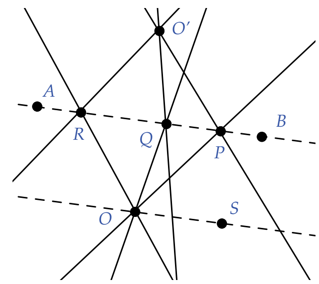
In Projective geometry, these are all equivalent figures under the appropriate projective transformation. Conic sections will be defined using properties of projective transformations. This might seem a bit strange, but there is a nice analog in Euclidean geometry where we can construct conic sections via isometries. This was covered briefly at the start of section 9.8. In this project we develop this idea in much more detail.

15.3.1 Euclidean Conic Sections Generated by Isometries

The *pencil of points with axis l* is the set of all points on l. The *pencil of lines with center O* is the set of all lines through point O. We have been using this terminology frequently in our study of Projective geometry, but in this section we will assume that pencils of points and lines consist solely of Euclidean points and lines.

We will be concerned with how pencils of points and lines are transformed under isometries. Consider how the pencil of lines with center O is transformed under the composition of two Euclidean isometries.

For example, let r be the reflection isometry across line $\overleftrightarrow{AB}$ as shown. Consider the pencil of lines with center O. Under the isometry r, one of these lines, say $\overleftrightarrow{OP}$, will map to $\overleftrightarrow{O'P}$, where P is the intersection point of the lines on $\overleftrightarrow{AB}$ and $r(O) = O'$. We say that $\overleftrightarrow{OP}$ and $\overleftrightarrow{O'P}$ are *corresponding* lines under the transformation r.



Now, consider the set of all intersection points of corresponding lines. These would include points P, Q, and R as shown above. This set of points is the *locus* of the points of intersection of corresponding lines of two pencils (the one at O and the one at O') that are related by the reflection r.

Definition 15.15. *A set of points is called a* locus of points *if each point in the set satisfies some geometric condition. A point is a member of the locus of points if and only if it satisfies the condition.*

In the previous example, a point is in the locus of points if it satisfies the condition that it is a point of intersection of corresponding lines of the two pencils. Clearly, this set of points is the line $\overleftrightarrow{AB}$. There is one unique line which does not generate an element in this locus – the line $\overleftrightarrow{OS}$ which is parallel to $\overleftrightarrow{AB}$ at O. However, if we consider this example in the extended Euclidean plane, with points at infinity attached, then $\overleftrightarrow{OS}$ and $r(\overleftrightarrow{OS})$ will intersect at a point at infinity, which we would then have to add to the locus of points.

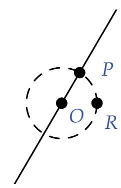
Exercise 15.3.1. *Show that the locus of points of intersection of corresponding lines of two pencils that are related by a translation T consists of* *no ordinary Euclidean points, but consists entirely of points at infinity. We consider this the* line at infinity*.*

So far we have constructed loci of points under reflections and translations and these loci turn out to be lines. While not one of the standard conic sections, a line is still a section of the cone, a so-called *degenerate* conic section.

Now let's see how we can construct a particular conic section, the circle, as a locus of points.

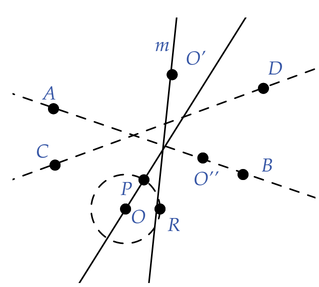
The Circle as a Locus under Two Reflections

Start up your dynamic geometry software. Construct point O to serve as the center for a pencil of lines. Construct a small circle centered at O with radius point R.



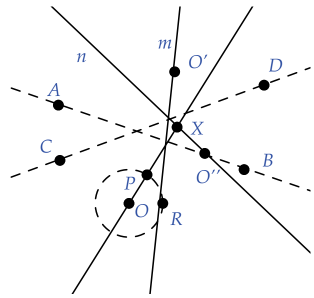
For ease of viewing, we have changed the drawing style of the circle to be dashed. To do this choose. Attach point P to the circle and construct $\overleftrightarrow{OP}$. By moving point P we can generate all of the different possible lines in the pencil of lines at O.

Next we create two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ to serve as two lines of reflection. Make these dashed lines and set $\overleftrightarrow{AB}$ as a line of reflection.



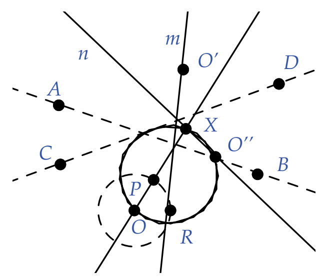
Select $\overleftrightarrow{AB}$ and then reflect this line. Line m is the reflected line. Select O and reflect this as well, creating point O'.

Now, we will carry out the second reflection. Set $\overleftrightarrow{CD}$ as a line of reflection and reflect m across this line to create line n. Also, reflect O' to point O''. Let f be the isometry that is the composition of these two reflections. Under f line $\overleftrightarrow{OP}$, from the pencil of lines at O, is mapped to line n, from the pencil of lines at O''. Construct the intersection X of $\overleftrightarrow{OP}$ and n.



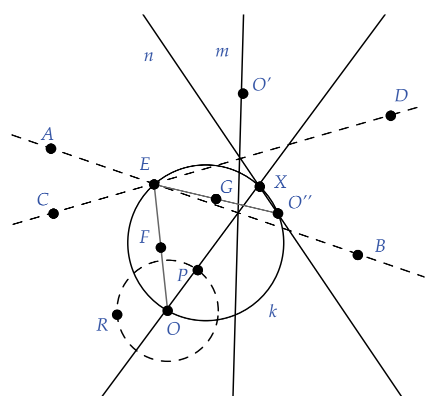
Now, consider the locus of points of intersection of corresponding lines of the two pencils at O and O''. This will be precisely the set of points generated from positions that X takes on as we move point P. Move point P around the circle at O and see what happens to X. It appears to sweep out a circle!

Consult the documentation on constructing loci for your dynamic geometry software. Then, construct the locus of points for X based on positions for P on the circle. It certainly appears that the locus is a circle.

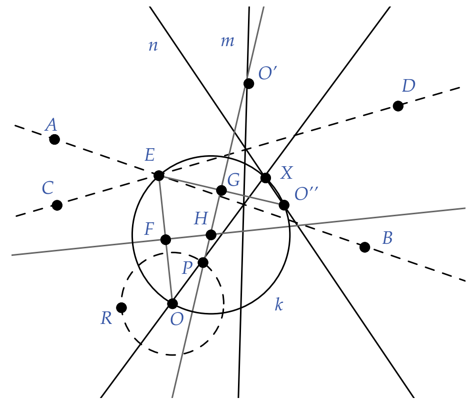


Let's prove that the locus is actually a circle.

Undo the locus construction. Construct the point of intersection E of the two lines of reflection. This point will be in the desired locus. Next construct $\overline{EO}$ and $\overline{EO''}$. Select these two segments and construct the midpoints (F and G) of each.



Next construct the perpendicular to $\overline{EO}$ at F. Likewise construct the perpendicular to $\overline{EO''}$ at G, and then construct H, the intersection point of these two perpendiculars. Finally, construct the circle k with center H and radius point E.



We know from our work in Project 2.2 that the perpendicular bisectors of the sides of a triangle intersect at a common point called the *circumcenter* of the triangle. The circle k constructed above, with center at the circumcenter and radius point E, will be the circumscribed circle of the triangle, and thus must pass through O and O''.

Exercise 15.3.2. *Show that the measure of $\angle OEO''$ is twice the measure of $\angle AEC$ and that the measure of $\angle OEO''$ is one-half the measure of $\angle OHO''$.*

Measure $\angle OEO''$, $\angle AEC$, and $\angle OHO''$ to verify the angle relationships stated in the previous exercise.

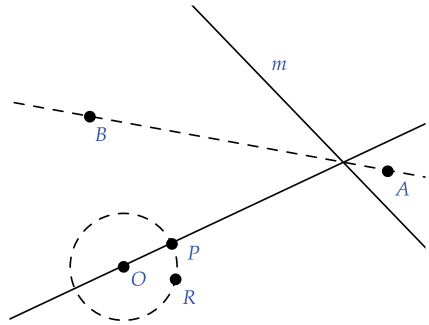
From Exercise 5.4.7 in Chapter 5, we know that the measure of the vertical angle at X made by $\overleftrightarrow{OP}$ and n must equal the angle of rotation. Thus, the measure of $\angle OXO''$ equals the measure of $\angle OEO''$, and thus the the measure of $\angle OXO''$ is one-half the measure of central angle $\angle OHO''$. By Theorem 2.42 the point X must be on the circle k. We therefore have proved the following:

Theorem 15.16. *Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ be intersecting lines in the plane. Let r_1 and r_2 be reflections across these lines. Let $O'' = r_2(r_1(O))$. Then, the locus of points of intersection of corresponding lines of the two pencils at O and O'' forms a circle.*

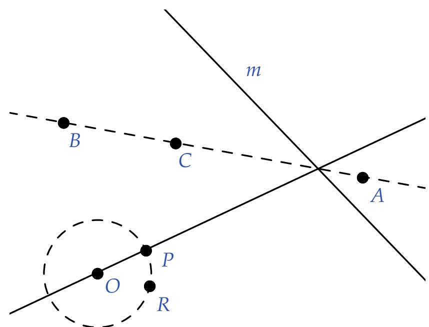
Glide Reflections and Hyperbolas

We have considered the locus of points of intersection of corresponding lines of two pencils of lines under three types of isometries - reflections, translations, and rotations. What happens if we use a glide reflection as our isometry? A glide reflection is composed of a reflection across a line followed by a translation in a direction parallel to a line.

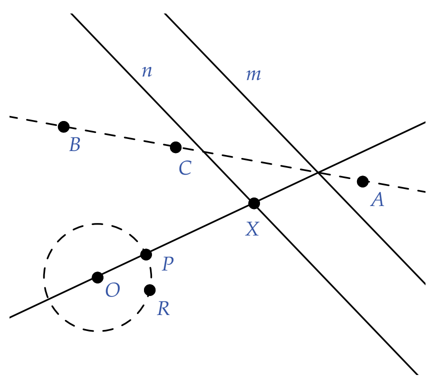
As in the last exploration, construct a point O to serve as the center for a pencil of lines. Construct a small circle centered at O with radius point R. Attach point P to the circle and construct $\overleftrightarrow{OP}$. Create $\overleftrightarrow{AB}$, set $\overleftrightarrow{AB}$ as a line of reflection, and reflect $\overleftrightarrow{OP}$ to get line m.



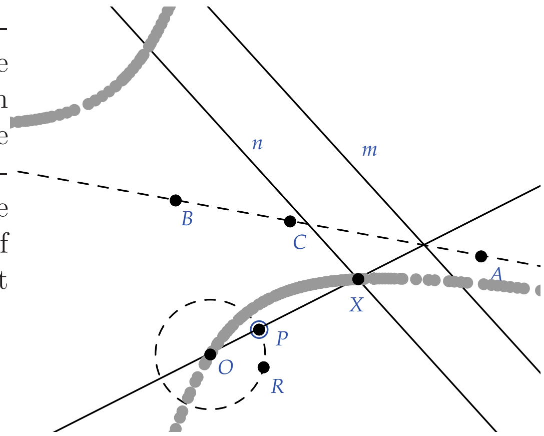
Next attach a point C to $\overleftrightarrow{AB}$ and define the translation vector $\overrightarrow{CB}$.



We have defined a translation that will be parallel to $\overleftrightarrow{AB}$ and will serve as the translation for our glide reflection. Select line m and translate it, creating line n. Construct the intersection point X of line n with $\overleftrightarrow{OP}$.



The locus of points of intersection of corresponding lines will be the set of points generated from positions that X takes on as we move point P. Construct this locus and move point P around the circle at O to generate a trace of the locus. It appears to sweep out a hyperbola!



To see why the locus is a hyperbola, we will consider a coordinate representation of the glide reflection and the pencil of points. We can assume the line of reflection is the x-axis. Then, the glide can be written as $g(x, y) = (x + v, -y)$, for some non-zero v. We can assume that we have chosen the origin so that the pencil of points can be represented as the set of lines through $O = (0, b)$. These lines have the form $y = mx - b$, or $(x, mx - b)$ as ordered pairs. Under the glide, these lines go to $(x + v, -mx + b)$. A shift in the x direction yields the transformed lines as $(x, -m(x - v) + b) = (x, -mx + mv + b)$.

To find the locus point, we find the intersection of these lines. Thus, we have $mx - b = -mx + mv + b$, or $2mx = mv + 2b$. Thus, $x = \frac{v}{2} + \frac{b}{m}$. Solving for y in $y = mx - b$, we get $y = \frac{mv}{2}$. Let $a = \frac{v}{2}$, which will be a constant. Then,

$$y(x - a) = (\frac{mv}{2})(\frac{v}{2} + \frac{b}{m} - \frac{v}{2}) = (\frac{mv}{2})(\frac{b}{m}) = \frac{vb}{2}.$$

We conclude that the coordinates for X satisfy an equation of the form $y(x-a) = c$, where a and c are constants. This is the coordinate equation for a hyperbola.

Envelopes of Lines for Pencils of Points

So far we have looked at how pencils of lines can be used to create locus sets of points that generate conic sections. In the spirit of duality, we now look at how pencils of points can lead to *envelope* sets of lines.

Definition 15.16. *A set of lines is called an* envelope of lines *if each line in the set satisfies some geometric condition. A line is a member of the envelope of lines if and only if it satisfies the condition.*

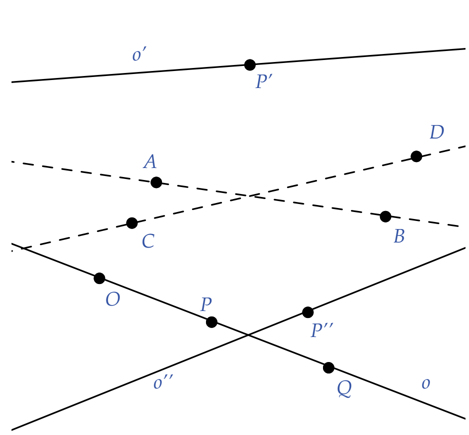
The envelope of a set of lines is dual to the concept of a locus of points. We describe the *curve* associated to a locus of points as the figure created by the points. Earlier in this section, we constructed loci that generated circles and hyperbolas. In this section, we will consider curves generated by envelopes of lines. To be precise, we will say that a curve c is generated by an envelope of lines if, for every point P on c, we have that there is a tangent line to the curve at P and this tangent line is a member of the envelope of lines.

In the last section, we looked at loci of points generated from pencils of lines, where the lines are transformed by compositions of two isometries. In this section we look at envelopes of lines generated from corresponding points of pencils of points, where the points are transformed by compositions of two isometries.

Recall that the pencil of points on a line l is just the set of all points on l. Starting with a pencil of points, we consider how that pencil transforms under two Euclidean isometries and then look at the envelope of lines constructed from corresponding points.

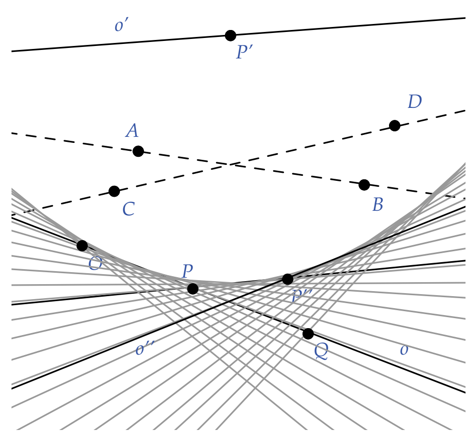
We know from our work in Chapter 5 that the composition of two isometries will be equivalent to a rotation, a reflection, a translation, or a glide reflection. In our analysis of envelopes of lines, we will not consider all possible compositions of two isometries, but will consider only one case as an example – the case where two reflections can be composed to create a rotation. For the reflections to create a rotation, the reflection lines must intersect.

Construct $o = \overleftrightarrow{OQ}$ to serve as the "center" for a pencil of points. Attach a point P to o. P serves as a representative of the pencil of points on o. Create $\overleftrightarrow{AB}$, set $\overleftrightarrow{AB}$ as a line of reflection, and reflect P to get P' and line o to get o'. Then, create $\overleftrightarrow{CD}$, set $\overleftrightarrow{CD}$ as a line of reflection, and reflect P' to get P'' and line o' to get o''.



Points P and P'' are corresponding points of the two pencils of lines from o and o''. In the previous section of this project, we considered the locus of points of intersection of corresponding lines of two pencils of lines. The dual of this will be to consider the envelope of lines created from corresponding points of two pencils of points.

Create the line $\overleftrightarrow{PP''}$ based on corresponding points P and P''. To construct the envelope of lines, construct the locus of lines generated from $\overleftrightarrow{PP''}$ as P moves along line o. It appears that the envelope of lines generates or *envelops* another conic section —the parabola.



By using properties of rotations from section 5.4, and triangle congruence results, one can show that the envelope of lines in this case is the envelope of tangent lines to a parabola that has focus point at the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. You can do the proof for extra credit for this project.

15.3.2 Projective Conic Sections Generated by Projectivities

In this section we will see how conics in projective geometry are constructed using properties of projective transformations. The construction will be very similar to our work in the preceding parts of this project – the only difference will be the type of transformations used in the construction.

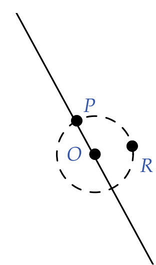
In section 9.4 we saw that the projective transformations that worked similarly to isometries in Euclidean geometry were the *projectivities*. A *projectivity* is a transformation that can be expressed as a finite composition of perspectivities. By Theorem 9.12 we know that any projectivity is equivalent to the composition of at most two perspectivities.

Proceeding by analogy to the work we did earlier in this project on conics in Euclidean geometry, we will consider the locus of points that is generated from corresponding points of pencils of lines, when such pencils are transformed by the composition of two perspectivities.

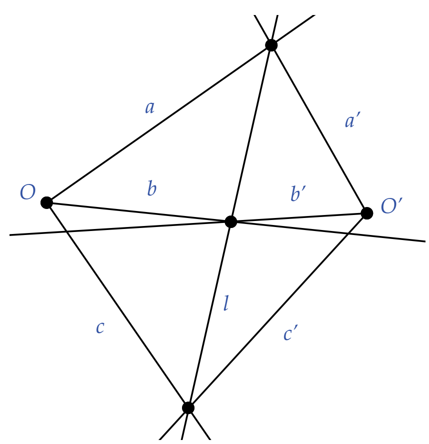
To explore these ideas we will use the Real model of Projective geometry, excluding the points at infinity. This model can be identified with the standard Euclidean plane. Our exploration can then be carried out using the Euclidean geometry features of your dynamic geometry software.

Our first exploration will be concerned with the effect of two perspectivities on a pencil of lines through a point O.

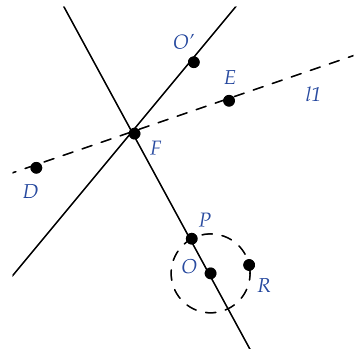
Create point O to serve as the center for our pencil of points. Construct a small circle at O with radius point R. Attach point P to the circle and construct $\overleftrightarrow{OP}$. As we move P we sweep out the pencil of lines at O.



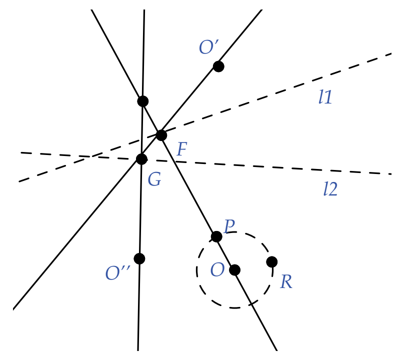
Recall that a perspectivity with axis l between a pencil of lines with center O and a pencil of points with center O' is a transformation that maps a line a through O to a line a' through O' such that the intersection of a and a' lies on l. We denote the perspectivity by $a \overset{l}{\barwedge} a'$.



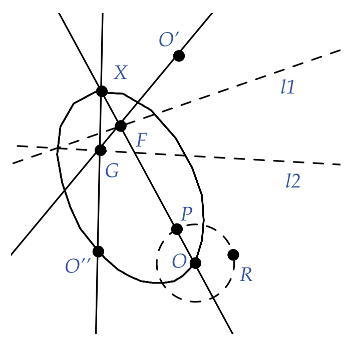
We need to create a sequence of two perspectivities. We create the first perspectivity by creating a point O' and an axis line $l1$. We have edited the style of $l1$ for ease of viewing. Construct the intersection point F of $\overleftrightarrow{OP}$ and $l1$. Then, construct $\overleftrightarrow{O'F}$. Then, $\overleftrightarrow{OP} \overset{l1}{\barwedge} \overleftrightarrow{O'F}$.



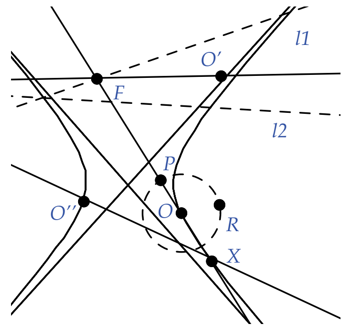
Define the second perspectivity by creating a point O'' and a second axis line $l2$. Construct the intersection point G of $\overleftrightarrow{O'F}$ and $l2$. Construct $\overleftrightarrow{O''G}$ so that $\overleftrightarrow{O'F} \overset{l2}{\barwedge} \overleftrightarrow{O''G}$. The corresponding lines under the composition of the two perspectivities will be $\overleftrightarrow{OP}$ and $\overleftrightarrow{O''G}$. Let X be the intersection of these lines.



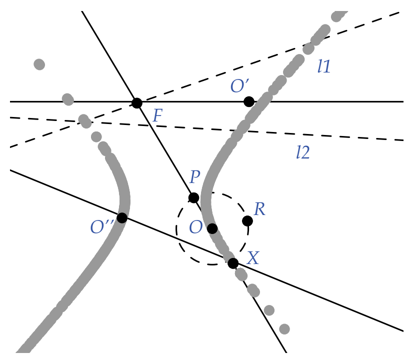
We now construct the locus of points of corresponding lines of the two pencils at O and O''. This will be the locus of positions for X as point P moves around the circle. In this case, the locus appears to be an ellipse!



If we move point O' around, the locus changes dramatically. In this position it looks a bit like a hyperbola, although there may be extraneous lines. This is because of the way some geometry programs connect the sample points of the locus.



For a better sense of the locus in this case, hide the locus and put a trace on point X. Move point P around the circle at O to generate the locus. It now does appear to sweep out a hyperbola!



Exercise 15.3.3. *Show that if* $l1$ *and* $l2$ *are the same line in the construction above, then the locus of points is a line.*

Exercise 15.3.4. *The line* $\overleftrightarrow{OO''}$ *must be a member of the pencil at* O *and the pencil at* O''*. Use this to show that points* O *and* O'' *must be on the locus of points.*

Our exploration provides strong evidence that the locus of points that is generated from corresponding points of pencils of lines, when such pencils are transformed by the composition of two perspectivities, yields a conic section. In section 9.8, we proved this result rigorously in the Real Projective Plane model of projective geometry.

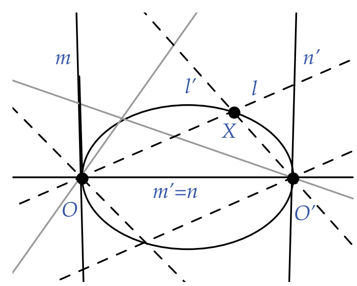
For your report give a careful and complete summary of your work done on this project.

15.4 CONICS AND TANGENTS

In Project 15.2.1 we saw that a conic section could be constructed from corresponding points of two pencils of lines, where one pencil is transformed into the other by the composition of two perspectivities, i.e by a projectivity. Given this result, it is reasonable to define conics in *axiomatic* projective geometry in a similar way. The axioms we assume will be P1-P7 as described earlier in this chapter. Our development in this section follows closely the work of W. T. Fishback in [9].

15.4.1 Point and Line Conics

Definition 15.17. *A* point conic *is the locus of points of intersection of corresponding lines of two pencils of lines, where the first pencil is transformed to the second by a projectivity. If the projectivity is equivalent to a single perspectivity, or if the centers of the pencils are the same point, the point conic will be called* singular*. Otherwise, the point conic is called* non-singular*.*



Here we have a non-degenerate point conic defined by a projectivity between pencils of lines at O and O'. For example, lines l and l' are projectively related, so their intersection point X is on the point conic. Note that for some line m through O, there will be a corresponding line m' through O' that meets m at O. Likewise, there is some line n' through O' that corresponds to line $n = m'$ through O.

From the figure, it appears that the centers of the two pencils lies on the point conic. This is always the case. Our definition of point conics splits the family of possible conics into two groups – the singular point conics and the non-singular point conics. We summarize below the major results on point (and line) conics. The proofs can be found in section 9.8 of Chapter 9.

Theorem 15.17. *The possible singular point conics include the following: the entire Projective plane, the set of points on two distinct lines, the set of points on a single line, and a single point.*

Singular point conics can consist of subsets that are lines. This is not the case for non-singular point conics.

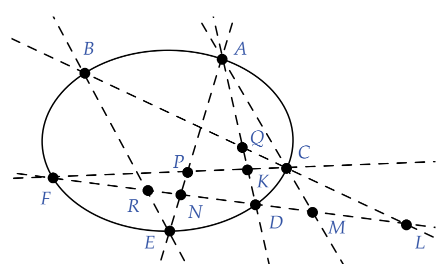
Theorem 15.18. *There are at most two distinct points of a non-singular point conic that lie on a given line.*

The next theorem tells us how to recognize when a set of points lies on a non-singular point conic.

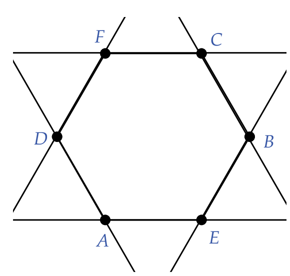
Theorem 15.19. *Let A, B, C, D, E, and F be distinct points such that no subset of three of the six points is collinear. Let P be the intersection of $\overleftrightarrow{AE}$ with $\overleftrightarrow{CF}$, Q the intersection of $\overleftrightarrow{AD}$ with $\overleftrightarrow{CB}$, and R the intersection of $\overleftrightarrow{BE}$ with $\overleftrightarrow{DF}$. Then C, D, E, and F are on a non-singular point conic determined by projectively related pencils of lines at A and B if and only if P, Q, and R are collinear.*

This theorem is a re-statement of one of the most famous theorems in geometry – Pascal's Theorem.

In the statement of Theorem 9.42 we looked at intersections of certain lines. If we take these lines and list them in the order where vertices match we have a six-sided figure —a hexagon. The hexagon is defined by $\overleftrightarrow{AE}$, $\overleftrightarrow{EB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CF}$, $\overleftrightarrow{FD}$, and $\overleftrightarrow{DA}$.



Of course, this hexagon is not made up of segments, but lines. How are the intersections chosen? Here we have "unwrapped" the hexagon into a more standard configuration., A quick check of the intersections from Theorem 9.42 shows that we are choosing *opposite sides* of the hexagon for intersections.



Pascal's Theorem in its classical form is as follows:

Theorem 15.20. *(Pascal's Theorem) If a hexagon is inscribed in a non-singular conic, then points of intersection of opposite sides are collinear.*

Pascal's Theorem leads directly to one of the key results in the theory of conics, first discovered by Jakob Steiner (1796-1963), who did pioneering work in the foundations of projective geometry.

Theorem 15.21. *(Steiner's Theorem) A non-singular point conic can be defined as the locus of points of intersection of two projectively related pencils of lines with centers at two arbitrarily chosen (distinct) points on the conic.*

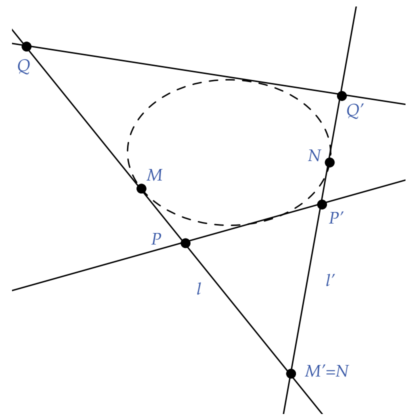
Steiner's Theorem leads to the following existence result for point conics.

Theorem 15.22. *Let A, B, C, D, and E be distinct points, no three of which are collinear. Then, there exists a unique non-singular point conic passing through these five points.*

The dual to a point conic is a line conic.

Definition 15.18. *A* line conic *is the envelope of lines defined by corresponding points of two pencils of points, where the first pencil is transformed to the second by a projectivity. If the projectivity is equivalent to a single perspectivity, or if the axes of the pencils are the same line, the line conic will be called* singular*. Otherwise, the line conic is called* non-singular*.*

Here we have a non-degenerate line conic defined by a projectivity between pencils of points on l and l'. For example, points P and P' are projectively related, so the line $\overleftrightarrow{PP'}$ is on the line conic. Note that for some point M on l, there will be a corresponding point M' on l' such that M' is on l. Likewise, there is some point N' on l' that corresponds to point $N = M'$ on l.



In the example above, we have shaded in an ellipse for reference purposes to help visualize the line conic. Here a few duals to the Theorems we have already proven for point conics.

Theorem 15.23. *The axes of the pencils that define a line conic always lie on the line conic.*

Theorem 15.24. *The possible singular line conics include the following: the entire Projective plane, the set of lines on two distinct points, the set of lines on a single point, and a single line.*

Theorem 15.25. *There are at most two distinct lines of a non-singular line conic that pass through a given point.*

Theorem 15.26. *(Dual to Pascal's Theorem) If a, b, c, d, e, and f are six distinct lines in a non-singular line conic, then the lines defined by joining opposite vertices are concurrent.*

A vertex is the intersection point of adjacent lines. For example, one vertex would be the intersection of a and b. The dual to Pascal's Theorem is known as Brianchon's Theorem, in honor of J. C. Brianchon (1785-1864).

15.4.2 Tangents

The case where a line intersects a non-singular point conic at a single point will be important enough to have its own definition.

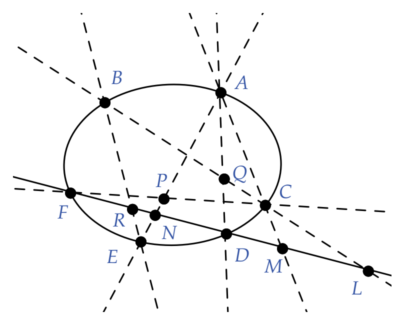
Definition 15.19. *A line is a* tangent line *to a non-singular point conic if it intersects the conic in exactly one point.*

The existence of tangents was proven in section 9.8.

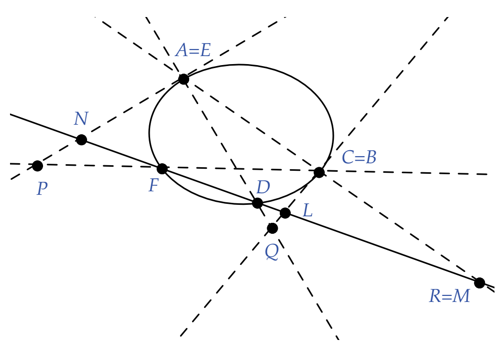
Theorem 15.27. *At each point on a non-singular point conic there is a unique tangent line.*

An interesting fact about tangents and Pascal's Theorem, is that we can replace pairs of edges with tangents and still have the conclusion of that theorem.

In the statement of Pascal's Theorem (or Theorem 9.42), we have hexagon $AEBCFD$ inscribed in a point conic. Consider what happens to the lines of this hexagon as we move point A to E and point B to C. The edges rotate into what seems to be the tangent lines at $A = E$ and $B = C$.



Here is a picture of what the lines and intersections look like after we replace the edges with tangent lines. That is, in the statement of the Theorem we replace $\overleftrightarrow{AE}$ with the tangent at A and $\overleftrightarrow{BC}$ with the tangent at B. If we carry this replacement of edge with tangent line throughout the proof of Theorem 15.19 the proof is still correct!



Theorem 15.28. *If $ABFD$ is a quadrangle inscribed in a non-singular point conic, let P be the intersection of the tangent at A with $\overleftrightarrow{BF}$, Q the intersection of the tangent at B and $\overleftrightarrow{AD}$, and R the intersection of $\overleftrightarrow{AB}$ with $\overleftrightarrow{DF}$. Then, P, Q, and R are collinear.*

If we let A move to D and B move to C we get the following result:

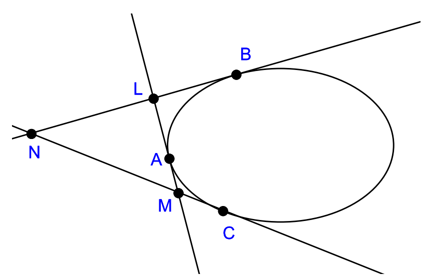
Theorem 15.29. *If $AEBF$ is a quadrangle inscribed in a non-singular point conic, let P be the intersection of $\overleftrightarrow{AE}$ with $\overleftrightarrow{BF}$, Q the intersection of the tangent at A with the tangent at B, and R the intersection of the tangent at E and the tangent at F. Then, P, Q, and R are collinear.*

The previous two results were proven in section 9.8. The following result describes a deep connection between line and point conics.

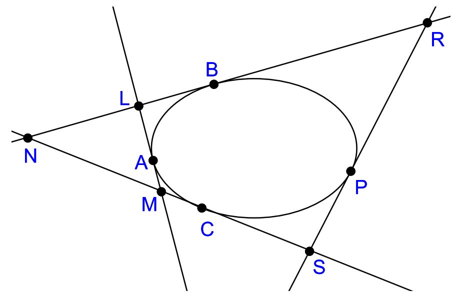
Theorem 15.30. *The set of tangents to a non-singular point conic form a line conic.*

Proof:

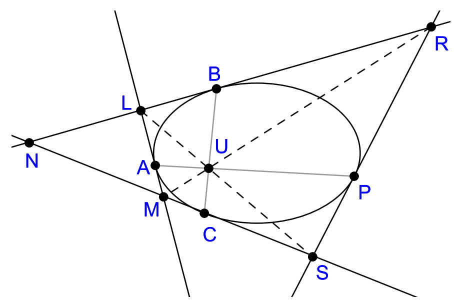
Let A, B, and C be points on the conic. Let the tangent at A and B meet at L, the tangent at A and C meet at M , and the tangent at B and C meet at N. Then, L, M, and N are not collinear, as the tangents at two distinct points on a conic cannot coincide (exercise).



By Corollary 9.7 there is a projectivity T from the pencil of points on the tangent at B to the pencil of points on the tangent at A with $T(N, L, B) = (C, M, N)$. The lines joining corresponding points under T are precisely the three tangent lines. Thus, we have the beginnings of a line conic, at least for these three pairs of points.



Now, let P be any other point on the conic. Let the tangent at P meet the tangent at B at R and the tangent at C at S. If we can show that $T(R) = S$, then we will have shown that the lines joining corresponding points of T are always tangent lines to the point conic. Thus, the line conic generated by T is the set of tangents to the point conic and our proof would be complete.



To show that $T(R) = S$, we will use some of the theorems we have proven on tangents and conics. We apply Theorem 15.29 to quadrangle $APBC$. Then, $U = AP \cdot BC$, L (intersection of tangent at A with tangent at B), and S (intersection of tangent at P with tangent at C) are collinear. If we apply the same theorem to quadrangle $APCB$, we get that U, M, and R are collinear.

Now, recall the projectivity version of Pappus's Theorem —Theorem 9.9. By this theorem, T defines a unique line called the axis of homology, which contains the intersections of the cross joins of all pairs of corresponding points. One cross join would be $LT(N) \cdot NT(L) = LC \cdot NM = C$ Another would be $LT(B) \cdot BT(L) = LN \cdot BM = B$. We conclude that BC is the axis of homology. Then, it must the the case that $LT(R) \cdot RT(L) = LT(R) \cdot RM$ must be on BC. Since RM intersects BC at U, we have that $LT(R)$ also intersects BC at U. But, LS intersects BC at U. Thus, $T(R)$ and S are both on LU and both on the tangent at C. Since two lines can only intersect in one point, we conclude that $T(R) = S$. □

Exercise 15.4.1. *Show that the tangents at two distinct points on a point conic cannot be the same line.*

Exercise 15.4.2. *Prove that if a triangle is inscribed in a point conic, then the tangents to the conic at the vertices of the triangle meet the opposite sides of the triangles at three points which must be collinear. [Hint: In Theorem 15.28, consider what happens as point F moves to become D.]*

Exercise 15.4.3. *Let P_1, P_2, P_3, P_4, and P_5 be five distinct points of a non-degenerate point conic. Let $Q = P_1P_2 \cdot P_4P_5$, $R = P_2P_3 \cdot P_5P_1$ and S be the intersection of P_3P_4 with the tangent to the conic at P_1. Show that Q, R, and S are collinear. [Hint: Consider Pascal's Theorem where one of the six points approaches another.]*

Exercise 15.4.4. *Show that Exercise 15.4.3 can be used to construct the tangent to a conic at a specific point. Describe the construction.*

Exercise 15.4.5. *Let P_1, P_2, P_3, and P_4 be four distinct points of a non-degenerate point conic. Let $Q = P_1P_2 \cdot P_3P_4$, $R = P_1P_3 \cdot P_2P_4$, S be the intersection of the tangent to the conic at P_1 with the tangent at P_4, and T be the intersection of the tangent at P_2 with the tangent at P_3. Show that Q, R, S, and T are collinear. [Hint: Consider Pascal's Theorem on $P_1P_{2a}P_{2b}P_4P_{3a}P_{3b}$ where we let the a, b points merge and also on $P_{1a}P_{1b}P_2P_{4a}P_{4b}P_3$ where we let the a, b points merge.]*

> **Definition 15.20.** *A point P is a* point of contact *of a non-degenerate line conic if it lies on exactly one line of the conic.*

Exercise 15.4.6. *What concept is the dual for a point of contact? Show that there exists one and only one point of contact on each line of a non-degenerate line conic.*

Exercise 15.4.7. *What is the dual theorem to Theorem 15.30?*