Theorem 1. *The language $L = \{a^k : k \text{ is a perfect square}\}$ is not regular.*

Proof. Assume for the sake of contradiction that L is regular. By the Pumping Lemma for Regular Languages (PLRL) there exists a positive constant n such that any string in L of length at least n can be "pumped." Consider the string $w = a^{n^2} \in L$. Since $|w| = n^2 \geq n$, the PLRL guarantees we can write w as xyz so that these three conditions hold:

(i) $|xy| \leq n$,

(ii) $y \neq \varepsilon$, and

(iii) $xy^iz \in L$ for all $i = 0, 1, 2, \ldots$.

Consider the string $s = xyyz$. By (iii) we have $s \in L$. Let $\ell = |y|$. Since $xyz = w = a^{n^2}$, it follows that $y = a^\ell$. Thus, $s = a^{n^2+\ell}$. However,

$$n^2 < n^2 + \ell < n^2 + 2n + 1 = (n+1)^2.$$

(The first inequality holds because of (ii) and $|y| = \ell$; the second inequality holds because $|y| = \ell$, and y is a substring of xy, and $|xy| \leq n$, and $n > 0$.) Hence, $|s| = n^2 + \ell$ is not a perfect square since it lies strictly between two consecutive perfect squares. Therefore, $a^{n^2+\ell} = s \notin L$. This gives the desired contradiction. □

---San Skulrattanakulchai