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Closure Properties of CFLs

Sipser Ch 2: Exercises 2.2, 2.15, 2.16, 2.18

The CFLs are closed under $\cup$, $\circ$, and *. They are not closed under $\cap$ or complement.

Theorem. *If P is a PDA and D is a DFA, then $L(P) \cap L(D)$ is a CFL.*

Proof. Construct a PDA that simultaneously simulates P and D. $\square$

Question Why does this construction fail when both P and D are PDA's?

Fix $\Sigma = \{a, b, c\}$. Let

$$E = \{\, w \in \Sigma^* : \#a(w) = \#b(w) = \#c(w) \,\}.$$

E is not a CFL. In proof, suppose it is. Then $F = E \cap L(a^*b^*c^*)$ is a CFL. But $F = \{\, a^n b^n c^n : n \geq 0 \,\}$ and we have already seen that F is not a CFL, a contradiction.

(i) The CFLs are not closed under intersection since E is the intersection of 2 CFLs:

$$E = \{\, w \in \Sigma^* : \#a(w) = \#b(w) \,\} \cap \{\, w \in \Sigma^* : \#b(w) = \#c(w) \,\}.$$

To show the RHS languages are CFL's, use PDAs.

(ii) The CFLs are not closed under complement since E is the complement of the CFL

$$\overline{E} = \{\, w \in \Sigma^* : \#a(w) \neq \#b(w) \,\} \cup \{\, w \in \Sigma^* : \#b(w) \neq \#c(w) \,\}.$$

To show the RHS languages are CFL's, note that $i \neq j$ means $i < j$ or $i > j$.