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Asymptotics

CLRS: Ch 3

Definitions

A real-valued function is said to be *defined on virtually all of the non-negative integers* $\mathbb{N}$ if it is not defined on only a finite number of them. For example, $f(n) = \lg\lg n$ is such a function since it is defined for all integers $n \geq 2$.
A function f that's defined on virtually all of $\mathbb{N}$ is said to be *eventually positive* if there exists $n_0 \in \mathbb{N}$ such that $f(n) > 0$ for all $n \geq n_0$. An example of such a function is $f(n) = n^2 - 4$; a non-example is the function $f(n) = -2n^2 + 100n + 50$.
The functions of resource (time & space) usage of algorithms are always defined on virtually all of $\mathbb{N}$, and are eventually positive.

Let f, g be such functions.
f is $O(g)$ (reads "f is of order at most g" or "f is big-Oh of g") means there exist positive number c, and non-negative integer n_0 such that $f(n) \leq cg(n)$ for all $n \geq n_0$.

Examples

$1000n - 2000$ is $O(n)$.
$n^3 + 100n^2$ is $O(n^3)$.
$n + \sqrt{n}$ is $O(n)$; $5\sqrt{n} + 500n^{1/3}$ is $O(n^{1/2})$.
$3 + \sin n$ is $O(1)$.
$\log_a n$ is $O(\log_b n)$ for any constants $a, b > 1$. (So, bases may be dropped.)
$20 \log n$ is $O(\log^2 n)$.
2^{n+1} is $O(2^n)$; 2^{2n} is not $O(2^n)$. (CLRS: Ex 3.1-4)

Helpful rules

Let functions f, F, g, G be given such that f is $O(F)$, and g is $O(G)$.

Rule of Products: fg is $O(FG)$.
E.g., $(1000n)(0.5n^2 + 50n^{3/2})$ is $O(n^3)$.
Rule of Exponents: f^k is $O(F^k)$ for any constant $k > 0$.

E.g., $(n+10)^2$ is $O(n^2)$; $\sqrt{n^2+100n}$ is $O(n)$.

Rule of Sums: $f+g$ is $O(\max\{F,G\})$.

E.g., $10n^4+1000n^3$ is $O(n^4)$; $(n^3/10+100n^2)(\log^3 n+50\log^2 n+\log n)^4$ is $O(n^3\log^{12} n)$.

Other asymptotic notions

We express upper bounds with $O(\cdot)$. Similarly, we can express lower bounds with $\Omega(\cdot)$ and equality with $\Theta(\cdot)$.

f is $\Omega(g)$ means there exist positive number c, and non-negative integer n_0 such that $f(n) \geq cg(n)$ for all $n \geq n_0$.

Examples

$n\sqrt{n}$ is $\Omega(n)$; $n/100$ is $\Omega(n)$.

f is $\Theta(g)$ means there exist positive numbers c_1, c_2, and non-negative integer n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq n_0$.

Examples:

$100n$ is $\Theta(n)$;

$23n^3+1000n^2-10$ is $\Theta(n^3)$.

The following two notations $o(\cdot)$ and $\omega(\cdot)$ behave similarly to $\prec$ and $\succ$, respectively.

f is $o(g)$ means $\lim_{n\to\infty} f(n)/g(n) = 0$.

f is $\omega(g)$ means $\lim_{n\to\infty} g(n)/f(n) = 0$.

Exercises

1. Show that Ω and Θ satisfy the same rules as for O given previously.

2. Show that f is $\Theta(g)$ if and only if f is $O(g)$ and f is $\Omega(g)$.

3. Show that O, Ω, and Θ behave similarly to $\preceq$, $\succeq$, and $\equiv$ respectively (i.e., "partial order," "partial order," and "equivalence relation" respectively).

4. Show that o and ω behave similarly to $\prec$ and $\succ$ respectively, i.e., they are "strict partial order" relations.

5. (Polynomials) Fix a positive integer d, positive number a_d and real numbers $a_0, a_1, \ldots, a_{d-1}$. Show that $\sum_{i=0}^{d} a_i n^i$ is $\Theta(n^d)$. For example, $n^{10}-100n^9+1010n^4-200$ is $\Theta(n^{10})$.

6. $\sum_{i=1}^{n} i^k$ is $\Theta(n^{k+1})$ whenever k is any fixed, positive constant.

7. $\sum_{i=0}^{n} a^i = \begin{cases} \Theta(1) & \text{if } a < 1 \\ \Theta(n) & \text{if } a = 1 \\ \Theta(a^{n+1}) & \text{if } a > 1 \end{cases}$

8. $\log n!$ is $\Theta(n \log n)$.

Notes

1. Definition of $O, \Omega, \Theta, o, \omega$ can readily be generalized to functions of > 1 variable, and/or functions whose domain is the set of reals.

2. "Splitting" technique is useful for proving asymptotics.

3. People also write $f = O(g)$, $f = \Theta(g)$, etc. In such cases, $O(g)$ and $\Theta(g)$ should be interpreted as sets, and $=$ should be interpreted as $\in$. When asymptotics appear in a formula, they need to be properly intrepreted correctly as well. Moreover, their meanings depend on whether they appear on the left or right side of the equality sign! (See CLRS p49–50).