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Review of Graphs & Digraphs

CLRS: Appendix B.4

A graph (digraph) $G = (V, E)$ consists of a finite set V of *vertices* or *nodes* and a set E of unordered (ordered) pairs of vertices called *edges*. In all our discussion about graphs n denotes the number of vertices and m denotes the number of edges.

Vertex u in an undirected graph is *adjacent to* or is a *neighbor* of vertex v if $e = \{u, v\}$ is an edge, in which case we also say that edge e is *incident to* or *incident with* vertex u.

Let $e = (u, v)$ be an edge in a digraph. Vertex u is *adjacent to* vertex v, and vertex v is *adjacent from* vertex u. Edge e *leaves* or *is incident from* u, and *enters* or *is incident to* v.

A graph (digraph) is called *complete* if it contains all possible edges, i.e., any vertex is adjacent to every other vertex. A complete graph has $m = n(n-1)/2$. A complete digraph has $m = n^2$.

A vertex is said to be *isolated* if it is on no edge. We normally assume our graphs contain no isolated vertices since in most applications isolated vertices do not affect the main algorithm. This implies $m \geq n/2$.

In general, m is $\Omega(n)$ and m is $O(n^2)$.

Graphs with $\Theta(n)$ edges are called *sparse*, those with $\Theta(n^2)$ edges are called *dense*. Examples of sparse graphs are the grid graphs and planar graphs.

Graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A *path* P is a sequence of vertices $v_0, v_1, \ldots, v_k$ for some $k \geq 0$ with v_i adjacent to v_{i+1} for all $0 \leq i < k$. The length of the path P is k. A path is *simple* if it repeats no vertex.

The *degree* of a vertex in an undirected graph is the number of edges incident to it. The *indegree* (*outdegree*) of a vertex in a digraph is the number of edges that enters (leaves) it.

Lemma. *In an undirected graph the degrees sum to $2m$. In a digraph the indegrees (outdegrees) sum to m.* □

An undirected graph is *connected* if for any two vertices there is a path from one vertex to the other. A *connected component* of a graph is a (vertex- and edge-) maximal connected subgraph.

Examples: The 15-tiles puzzle has $16! \approx 2 \times 10^{13}$ vertices. It has 2 connected components, each with $\approx 10^{13}$ vertices. ...

A *forest* is a graph with no cycle. A *tree* is a connected forest. Any tree can be *rooted* by making some vertex r distinquished. Vertex r then becomes the root.

Every connected undirected graph G has a *spanning tree*, i.e., a subgraph without cycle and containing every vertex of G.

Examples: ...

Graph Operations

Deleting edge e from (di)graph G means forming the (di)graph $G-e$ having all edges of G except e

Examples: ...

Deleting vertex v from graph (digraph) G means forming the graph $G-v$ having all edges of G except v and all edges of G except those incident to (to or from) v.

Examples: ...

Contracting a set of vertices S means forming the graph (digraph) G/S where the vertices S are replaced by a new vertex Σ, adjacent to (to or from) every neighbor of S.

Examples: ...

Contracting an edge e means contracting the two ends of e.

Notes

1. *Multigraphs* and *multidigraphs* are like graphs and digraphs except that multiple edges joining the same pair of vertices are allowed. All graph-theoretic definitions made here can be straightforwardly generalized to multigraphs and multidigraphs.

2. An edge from any vertex v to itself is called a *self-loop*. Self-loops are generally not allowed in a graph or digraph, but are usually allowed in multigraphs and multidigraphs. CLRS allows digraphs to have self-loops.

3. There are many different terminologies in current use in Graph Theory literature. Worse still, the same term is often used to mean different things by different authors. Make sure you know what your author means!